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Neo Cobalancing Numbers

Ahmet Tekcan and Özlem Duru

Bursa Uludag University, Faculty of Science,
Department of Mathematics, Bursa-Türkiye
Email ID: tekcan@uludag.edu.tr,
Email ID: 502511003@ogr.uludag.edu.tr

Abstract

In this work, we defined neo cobalancing numbers and then determined the general terms of them in terms of balancing and Lucas-balancing or cobalancing and Lucas-cobalancing numbers. We also deduced some algebraic relations on Binet formulas, relationship with balancing, Pell, Pell-Lucas, triangular, square triangular numbers, sums of first n-terms of them, Cassini and Catalan identities.
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1. Introduction.

A positive integer n is called a balancing number ([2]) if the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{1.1}$$

holds for some positive integer r which is called balancer. From (1.1)

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 1}}{2}. \tag{1.2}$$

Though the definition of balancing numbers suggests that no balancing number should be less than 2. But from (1.2), Behera and Panda noted that $8(0)^2+1=1$ and $8(1)^2+1=3^2$ are perfect squares. So they accepted 0 and 1 to be balancing numbers.

A positive integer n is called a cobalancing number ([16]) if the Diophantine equation

$$1+2+\cdots+n=(n+1)+(n+2)+\cdots+(n+r) \tag{1.3}$$

holds for some positive integer r which is called cobalancer. Form (1.3)

$$r=\frac{-2n-1+\sqrt{8n^2+8n+1}}{2}. \tag{1.4}$$

From definition cobalancing numbers should be positive. But form (1.4), Panda and Ray noted that $8(0)^2+8(0)+1=1$ is a perfect square. So they accepted 0 to be a cobalancing number just like Behera and Panda accepted 0 and 1 to be balancing numbers.

Let B_n denote the nth balancing number and let b_n denote the nth cobalancing number. Then $B_0=0, B_1=1, B_2=6, B_{n+1}=6B_n-B_{n-1}$ and $b_0=b_1=0, b_2=2, b_{n+1}=6b_n-b_{n-1}+2$ for $n\geq 2$. Also from (1.2), B_n is a balancing number if and only if $8B_n^2+1$ is a perfect square and from (1.4), b_n is a cobalancing number if and only if $8b_n^2+8b_n+1$ is a perfect square. Thus

$$C_n=\sqrt{8B_n^2+1} \quad \text{and} \quad c_n=\sqrt{8b_n^2+8b_n+1}$$

are integers which are called the nth Lucas-balancing and the nth Lucas-cobalancing number, respectively.

Ray proved ([20]) that Binet formulas are $B_n=\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}, b_n=\frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}}-\frac{1}{2}, C_n=\frac{\alpha^{2n}+\beta^{2n}}{2}$ and $c_n=\frac{\alpha^{2n-1}+\beta^{2n-1}}{2}$ for $n\geq 1$, where $\alpha=1+\sqrt{2}$ and $\beta=1-\sqrt{2}$ which are the roots of the characteristic equation of Pell P_n and Pell-Lucas numbers Q_n (Note that $B_{-n}=-B_n, b_{-n}=b_{n+1}, C_{-n}=C_n, c_{-n}=-c_{n+1}, P_{-n}=(-1)^{n+1}P_n$ and $Q_{-n}=(-1)^nQ_n$) (see also [13, 15]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [11], Liptai proved that there is no Fibonacci balancing number except 1 and in [12] he proved that there is no Lucas-balancing number. In [23], Szalay considered the same problem and obtained some nice results by a different method. In [9], Kovács, Liptai, Olajos extended the concept of balancing numbers to the (a,b)-balancing numbers. Let $a>0$ and $b\geq 0$ be coprime integers. If

$$(a+b)+\cdots+(a(n-1)+b)=(a(n+1)+b)+\cdots+(a(n+r)+b)$$

for some positive integers n and r, then $an+b$ is an (a,b)-balancing number. The sequence of (a,b)-balancing numbers is denoted by $B_m^{(a,b)}$ for $m \geq 1$. In [10], the authors generalized the notion of balancing numbers to numbers defined as follows: Let $y,k,l \in \mathbb{Z}^+$ with $y \geq 4$. A positive integer x with $x \leq y-2$ is called a (k,l)-power numerical center for y if

$$1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l.$$

They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for (k,l)-power numerical centers. For positive integers k,x, let $\Pi_k(x) = x(x+1)\ldots(x+k-1)$. Then it was proved in [9] that the equation $B_m = \Pi_k(x)$ for fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2,3,4\}$ all solutions were determined. In [35] Tengely, considered the case $k=5$ and proved that this Diophantine equation has no integer solution for $m \geq 0$ and $x \in \mathbb{Z}$. In [18], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas-balancing numbers. In [14], Özdemir introduced a new non-commutative number system called hybrid numbers. In [3], Bród, Szynal-Liana and Włoch defined balancing and Lucas-balancing hybrid numbers and in [21], Rubajczyk and Szynal-Liana defined cobalancing and Lucas-cobalancing hybrid numbers. In [5], Dash, Ota, Dash defined t-balancing numbers. In [27], Tekcan and Aydın determined the general terms of t-balancing numbers. In [26], Tekcan and Erdem defined t-cobalancing numbers and determined the general terms them. In [17], Panda and Panda defined almost balancing numbers. In [19], Panda defined almost cobalancing numbers. In [31], Tekcan and Erdem determined the general terms of all almost balancing numbers and almost cobalancing numbers. In [24], Tekcan considered the sums and spectral norms of all almost balancing numbers and in [25], Tekcan derived some results on almost balancing numbers, triangular numbers and square triangular numbers. In [28, 29], Tekcan and Yıldız defined balcobalancing numbers and in [30, 32], they defined almost balcobalancing numbers. In [33], Tekcan and Akgüç defined almost neo cobalancing numbers and in [34], Tekcan defined almost neo balancing numbers.

2. Neo Cobalancing Numbers.

In [4], Chailangka and Pakapongpun defined neo balancing numbers. They said that an integer $n \geq 1$ is called a neo balancing number if the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n-1) + (n-0) + (n+1) + (n+2) + \cdots + (n+r) \tag{2.1}$$

holds for some integer $r \geq 1$ which is called neo balancer. By considering (2.1), we say that a positive integer n is called a neo cobalancing number if the Diophantine equation

$$1 + 2 + \cdots + n = (n-1) + (n-0) + (n+1) + (n+2) + \cdots + (n+r) \tag{2.2}$$

holds for some positive integer r which is called neo cobalancer.

From (2.2), we get

$$r = \frac{-2n - 1 + \sqrt{8n^2 - 8n + 9}}{2} \tag{2.3}$$

and

$$n = \frac{2r + 3 + \sqrt{8r^2 + 16r + 1}}{2}. \tag{2.4}$$

From (2.3), we note that $8(1)^2 - 8(1) + 9 = 3^2$ and $8(2)^2 - 8(2) + 9 = 5^2$ are perfect squares. But in this case $r = 0$ and $r = 0$, respectively. Nevertheless, we accept 1 and 2 to be a neo cobalancing numbers just like Behera and Panda accepted 0 and 1 to be balancing numbers.

Let b_n^{neo} denote the nth neo cobalancing number and let r_n^{neo} denote the nth neo cobalancer. Then from (2.3), b_n^{neo} is a neo cobalancing number if and only if $8(b_n^{\text{neo}})^2 - 8b_n^{\text{neo}} + 9$ is a perfect square. Thus

$$c_n^{\text{neo}} = \sqrt{8(b_n^{\text{neo}})^2 - 8b_n^{\text{neo}} + 9} \tag{2.5}$$

is an integer which is called the nth neo Lucas-cobalancing number and from (2.4), r_n^{neo} is a neo cobalancer if and only if $8(r_n^{\text{neo}})^2 + 16r_n^{\text{neo}} + 1$ is a perfect square. Thus

$$cr_n^{\text{neo}} = \sqrt{8(r_n^{\text{neo}})^2 + 16r_n^{\text{neo}} + 1} \tag{2.6}$$

is an integer which is called the nth neo Lucas-cobalancer.

In order to determine the general terms of all neo cobalancing numbers, we have to determine the set of all integer solutions of some specific Pell equation (see [1, 8]). Indeed, since b_n^{neo} is a neo cobalancing number if and only if $8(b_n^{\text{neo}})^2 - 8b_n^{\text{neo}} + 9$ is a perfect square, we set

$$8(b_n^{\text{neo}})^2 - 8b_n^{\text{neo}} + 9 = y^2$$

for some integer $y \geq 1$. Then

$$2[(2b_n^{\text{neo}} - 1)^2 - 1] + 9 = y^2.$$

Taking

$$x = 2b_n^{\text{neo}} - 1,$$

we get the Pell equation

$$2x^2 - y^2 = -7. \tag{2.7}$$

Let Ω denotes the set of all integer solutions of (2.7). Then we can give the following theorem.

Theorem 2.1. *The set of all integer solutions of* (2.7) *is*

$$\Omega = \{(6B_{n-1} + C_{n-1}, 4B_{n-1} + 3C_{n-1}) : n \geq 1\} \cup \{(6B_n - C_n, -4B_n + 3C_n) : n \geq 1\}.$$

Proof. For the Pell equation in (2.7), the indefinite form is $F = (2, 0, -1)$ of discriminant $\Delta = 8$. So $\tau_8 = 3 + 2\sqrt{2}$. Thus Rep $= \{[\pm 1 \quad 3]\}$ and $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ (see [6, p. 121]). Here $[1 \quad 3]M^{n-1}$ generates all integer solutions (x_{2n-1}, y_{2n-1}) for $n \geq 1$ and $[-1 \quad 3]M^n$ generates all integer solutions (x_{2n}, y_{2n}) for $n \geq 1$. It can be easily seen that

$$M^n = \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix}$$

for $n \geq 1$. Thus

$$[x_{2n-1} \quad y_{2n-1}] = [1 \quad 3]M^{n-1} = [6B_{n-1} + C_{n-1} \quad 4B_{n-1} + 3C_{n-1}]$$
$$[x_{2n} \quad y_{2n}] = [-1 \quad 3]M^n = [6B_n - C_n \quad -4B_n + 3C_n].$$

So $\Omega = \{(6B_{n-1} + C_{n-1}, 4B_{n-1} + 3C_{n-1}) : n \geq 1\} \cup \{(6B_n - C_n, -4B_n + 3C_n) : n \geq 1\}$. □

From Theorem 2.1, we can give the following result.

Theorem 2.2. *The general terms of neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$b_{2n-1}^{neo} = \frac{6B_{n-1} + C_{n-1} + 1}{2}$$
$$b_{2n}^{neo} = \frac{6B_n - C_n + 1}{2}$$
$$c_{2n-1}^{neo} = 4B_{n-1} + 3C_{n-1}$$
$$c_{2n}^{neo} = -4B_n + 3C_n$$
$$r_{2n-1}^{neo} = -B_{n-1} + C_{n-1} - 1$$
$$r_{2n}^{neo} = -5B_n + 2C_n - 1$$

$$cr_{2n}^{neo} = 16B_n - 5C_n$$

for $n \geq 1$ *and*

$$cr_{2n-1}^{neo} = 8B_{n-1} - C_{n-1}$$

for $n \geq 2$.

Proof. Applying Theorem 2.1, we deduce that

$$b_{2n-1}^{\text{neo}} = \frac{6B_{n-1} + C_{n-1} + 1}{2}$$

since $x = 2b_n^{\text{neo}} - 1$. Thus from (2.5), we obtain

$$\begin{aligned}
c_{2n-1}^{\text{neo}} &= \sqrt{8(b_{2n-1}^{\text{neo}})^2 - 8b_{2n-1}^{\text{neo}} + 9} \\
&= \sqrt{8(\frac{6B_{n-1} + C_{n-1} + 1}{2})^2 - 8(\frac{6B_{n-1} + C_{n-1} + 1}{2}) + 9} \\
&= \sqrt{72B_{n-1}^2 + 24B_{n-1}C_{n-1} + 2(8B_{n-1}^2 + 1) + 7} \\
&= \sqrt{16B_{n-1}^2 + 24B_{n-1}C_{n-1} + 9C_{n-1}^2} \\
&= \sqrt{(4B_{n-1} + 3C_{n-1})^2} \\
&= 4B_{n-1} + 3C_{n-1}.
\end{aligned}$$

From (2.3), we deduce that

$$r_{2n-1}^{\text{neo}} = \frac{-2b_{2n-1}^{neo} - 1 + c_{2n-1}^{neo}}{2} = -B_{n-1} + C_{n-1} - 1.$$

Finally from (2.6), we get

$$\begin{aligned}
cr_{2n-1}^{\text{neo}} &= \sqrt{8(r_{2n-1}^{\text{neo}})^2 + 16r_{2n-1}^{\text{neo}} + 1} \\
&= \sqrt{8(-B_{n-1} + C_{n-1} - 1)^2 + 16(-B_{n-1} + C_{n-1} - 1) + 1} \\
&= \sqrt{8B_{n-1}^2 - 16B_{n-1}C_{n-1} + 8(8B_{n-1}^2 + 1) - 7} \\
&= \sqrt{64B_{n-1}^2 - 16B_{n-1}C_{n-1} + (8B_{n-1}^2 + 1)} \\
&= \sqrt{64B_{n-1}^2 - 16B_{n-1}C_{n-1} + C_{n-1}^2} \\
&= \sqrt{(8B_{n-1} - C_{n-1})^2} \\
&= 8B_{n-1} - C_{n-1}.
\end{aligned}$$

The others can be proved similarly. □

In Table 2.1, the first ten terms of all neo cobalancing numbers is given.

Table 2.1 Neo cobalancing numbers

i	b_n^{neo}	r_n^{neo}	c_n^{neo}	cr_n^{neo}
1	1	0	3	1
2	2	0	5	1
3	5	1	13	5
4	10	3	27	11
5	27	10	75	31
6	56	22	157	65
7	155	63	437	181
8	324	133	915	379
9	901	372	2547	1055
10	1886	780	5333	2209

Apart from Theorem 2.2, we can give the general terms of all neo cobalancing numbers in terms of cobalancing and Lucas-cobalancing numbers instead of balancing and Lucas-balancing numbers as follows.

Theorem 2.3. *The general terms of neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$\begin{aligned} b_{2n-1}^{neo} &= -b_n + c_n \\ b_{2n}^{neo} &= b_n + c_n + 1 \\ c_{2n-1}^{neo} &= 8b_n - c_n + 4 \\ c_{2n}^{neo} &= 8b_n + c_n + 4 \\ r_{2n-1}^{neo} &= \frac{10b_n - 3c_n + 3}{2} \\ r_{2n}^{neo} &= \frac{6b_n - c_n + 1}{2} \\ cr_{2n}^{neo} &= -4b_n + 3c_n - 2 \end{aligned}$$

for $n \geq 1$ *and*

$$cr_{2n-1}^{neo} = -12b_n + 5c_n - 6$$

for $n \geq 2$.

Proof. Applying Theorem 2.2, we deduce that

$$\begin{aligned}
b_{2n-1}^{\text{neo}} &= \frac{6B_{n-1}+C_{n-1}+1}{2} \\
&= \frac{6(\frac{\alpha^{2n-2}-\beta^{2n-2}}{4\sqrt{2}})+(\frac{\alpha^{2n-2}+\beta^{2n-2}}{2})+1}{2} \\
&= \frac{\alpha^{2n-2}(3+\sqrt{2})+\beta^{2n-2}(-3+\sqrt{2})+2\sqrt{2}}{4\sqrt{2}} \\
&= \alpha^{2n-2}(\frac{-\alpha+2\sqrt{2}\alpha}{4\sqrt{2}})+\beta^{2n-2}(\frac{\beta+2\sqrt{2}\beta}{4\sqrt{2}})+\frac{1}{2} \\
&= \alpha^{2n-2}(\frac{-\alpha}{4\sqrt{2}}+\frac{\alpha}{2})+\beta^{2n-2}(\frac{\beta}{4\sqrt{2}}+\frac{\beta}{2})+\frac{1}{2} \\
&= -(\frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}}-\frac{1}{2})+(\frac{\alpha^{2n-1}+\beta^{2n-1}}{2}) \\
&= -b_n+c_n.
\end{aligned}$$

The others can be proved similarly. □

3. Binet Formulas and Recurrence Relations.

Theorem 3.1. *Binet formulas for neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$\begin{aligned}
b_{2n-1}^{neo} &= \frac{(3+\sqrt{2})\alpha^{2n-2}-(3-\sqrt{2})\beta^{2n-2}+2\sqrt{2}}{4\sqrt{2}} \\
b_{2n}^{neo} &= \frac{(3-\sqrt{2})\alpha^{2n}-(3+\sqrt{2})\beta^{2n}+2\sqrt{2}}{4\sqrt{2}} \\
c_{2n-1}^{neo} &= \frac{(3+\sqrt{2})\alpha^{2n-2}+(3-\sqrt{2})\beta^{2n-2}}{2} \\
c_{2n}^{neo} &= \frac{(3-\sqrt{2})\alpha^{2n}+(3+\sqrt{2})\beta^{2n}}{2} \\
r_{2n-1}^{neo} &= \frac{(-1+2\sqrt{2})\alpha^{2n-2}+(1+2\sqrt{2})\beta^{2n-2}-4\sqrt{2}}{4\sqrt{2}} \\
r_{2n}^{neo} &= \frac{(-5+4\sqrt{2})\alpha^{2n}+(5+4\sqrt{2})\beta^{2n}-4\sqrt{2}}{4\sqrt{2}} \\
cr_{2n}^{neo} &= \frac{(-5+4\sqrt{2})\alpha^{2n}-(5+4\sqrt{2})\beta^{2n}}{2}
\end{aligned}$$

for $n \geq 1$ *and*

$$cr_{2n-1}^{neo} = \frac{(-1+2\sqrt{2})\alpha^{2n-2} - (1+2\sqrt{2})\beta^{2n-2}}{2}$$

for $n \geq 2$.

Proof. Recall that $B_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}$ and $C_n = \frac{\alpha^{2n}+\beta^{2n}}{2}$. Thus from Theorem 2.2, we easily get

$$\begin{aligned} b_{2n-1}^{\text{neo}} &= \frac{6B_{n-1} + C_{n-1} + 1}{2} \\ &= \frac{6(\frac{\alpha^{2n-2}-\beta^{2n-2}}{4\sqrt{2}}) + (\frac{\alpha^{2n-2}+\beta^{2n-2}}{2}) + 1}{2} \\ &= \frac{\alpha^{2n-2}(\frac{3}{2\sqrt{2}} + \frac{1}{2}) + \beta^{2n-2}(\frac{-3}{2\sqrt{2}} + \frac{1}{2}) + 1}{2} \\ &= \frac{(3+\sqrt{2})\alpha^{2n-2} - (3-\sqrt{2})\beta^{2n-2} + 2\sqrt{2}}{4\sqrt{2}}. \end{aligned}$$

The others can be proved similarly. □

Theorem 3.2. *Recurrence relations for neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$\begin{aligned} b_n^{neo} &= b_{n-1}^{neo} + 6b_{n-2}^{neo} - 6b_{n-3}^{neo} - b_{n-4}^{neo} + b_{n-5}^{neo} \\ c_n^{neo} &= c_{n-1}^{neo} + 6c_{n-2}^{neo} - 6c_{n-3}^{neo} - c_{n-4}^{neo} + c_{n-5}^{neo} \\ r_n^{neo} &= r_{n-1}^{neo} + 6r_{n-2}^{neo} - 6r_{n-3}^{neo} - r_{n-4}^{neo} + r_{n-5}^{neo} \end{aligned}$$

for $n \geq 6$ *and*

$$cr_n^{neo} = cr_{n-1}^{neo} + 6cr_{n-2}^{neo} - 6cr_{n-3}^{neo} - cr_{n-4}^{neo} + cr_{n-5}^{neo}$$

for $n \geq 7$.

Proof. Let n be even, say $n = 2k$ for some positive $k \geq 1$. Then from Theorem 3.1, we get

$$\begin{aligned} b_{n-1}^{\text{neo}} + 6b_{n-2}^{\text{neo}} - 6b_{n-3}^{\text{neo}} - b_{n-4}^{\text{neo}} + b_{n-5}^{\text{neo}} &= \frac{(3+\sqrt{2})\alpha^{2k-2} - (3-\sqrt{2})\beta^{2k-2} + 2\sqrt{2}}{4\sqrt{2}} \\ &\quad + 6\left(\frac{(3-\sqrt{2})\alpha^{2k-2} - (3+\sqrt{2})\beta^{2k-2} + 2\sqrt{2}}{4\sqrt{2}}\right) \\ &\quad - 6\left(\frac{(3+\sqrt{2})\alpha^{2k-4} - (3-\sqrt{2})\beta^{2k-4} + 2\sqrt{2}}{4\sqrt{2}}\right) \end{aligned}$$

$$
\begin{aligned}
&- \left(\frac{(3-\sqrt{2})\alpha^{2k-4} - (3+\sqrt{2})\beta^{2k-4} + 2\sqrt{2}}{4\sqrt{2}} \right) \\
&+ \left(\frac{(3+\sqrt{2})\alpha^{2k-6} - (3-\sqrt{2})\beta^{2k-6} + 2\sqrt{2}}{4\sqrt{2}} \right) \\
&= \left(\frac{(21-5\sqrt{2})\alpha^{2k-2} - (21+5\sqrt{2})\beta^{2k-2} + 14\sqrt{2}}{4\sqrt{2}} \right) \\
&- \left(\frac{(21+5\sqrt{2})\alpha^{2k-4} - (21-5\sqrt{2})\beta^{2k-4} + 14\sqrt{2}}{4\sqrt{2}} \right) \\
&+ \left(\frac{(3+\sqrt{2})\alpha^{2k-6} - (3-\sqrt{2})\beta^{2k-6} + 2\sqrt{2}}{4\sqrt{2}} \right) \\
&= \frac{(3-\sqrt{2})\alpha^{2k} - (3+\sqrt{2})\beta^{2k} + 2\sqrt{2}}{4\sqrt{2}} \\
&= b_{2k}^{\mathrm{neo}} \\
&= b_n^{\mathrm{neo}}.
\end{aligned}
$$

The others can be proved similarly. □

Apart from Theorem 3.2, we can give the following result which can be proved similarly.

Theorem 3.3. *For neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers, we have*

$$
\begin{aligned}
b_{2n+1}^{neo} &= \frac{7b_{2n}^{neo} - 3b_{2n-1}^{neo} - 1}{2} \\
c_{2n+1}^{neo} &= \frac{7c_{2n}^{neo} - 3c_{2n-1}^{neo}}{2} \\
r_{2n+1}^{neo} &= \frac{7r_{2n}^{neo} - 3r_{2n-1}^{neo} + 2}{2}
\end{aligned}
$$

for $n \geq 1$ *and*

$$
\begin{aligned}
b_{2n}^{neo} &= \frac{7b_{2n-1}^{neo} - 2b_{2n-2}^{neo} - 1}{3} \\
c_{2n}^{neo} &= \frac{7c_{2n-1}^{neo} - 2c_{2n-2}^{neo}}{3} \\
r_{2n}^{neo} &= \frac{7r_{2n-1}^{neo} - 2r_{2n-2}^{neo} + 2}{3} \\
cr_{2n}^{neo} &= \frac{7cr_{2n-1}^{neo} - 2cr_{2n-2}^{neo}}{3}
\end{aligned}
$$

$$cr_{2n+1}^{neo} = \frac{7cr_{2n}^{neo} - 3cr_{2n-1}^{neo}}{2}$$

for $n \geq 2$.

4. Sums of Neo Cobalancing Numbers.

Recall that the sum of first n-terms of all balancing numbers can be given in terms of same balancing numbers, that is,

$$\sum_{i=1}^{n} B_i = \frac{5B_n - B_{n-1} - 1}{4},\ n \geq 1 \qquad \sum_{i=1}^{n} b_i = \frac{5b_n - b_{n-1} + 2 - 2n}{4},\ n \geq 1$$

$$\sum_{i=1}^{n} C_i = \frac{5C_n - C_{n-1} - 2}{4},\ n \geq 1 \qquad \sum_{i=1}^{n} c_i = \frac{5c_n - c_{n-1} - 2}{4}\ , n \geq 2.$$

Similarly we can give the following theorem.

Theorem 4.1. *The sums of first n-terms of neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$\sum_{i=1}^{n} b_i^{neo} = \frac{5b_n^{neo} + 5b_{n-1}^{neo} - b_{n-2}^{neo} - b_{n-3}^{neo} + 2n - 8}{4}$$

$$\sum_{i=1}^{n} c_i^{neo} = \frac{5c_n^{neo} + 5c_{n-1}^{neo} - c_{n-2}^{neo} - c_{n-3}^{neo}}{4}$$

$$\sum_{i=1}^{n} r_i^{neo} = \frac{5r_n^{neo} + 5r_{n-1}^{neo} - r_{n-2}^{neo} - r_{n-3}^{neo} - 4n + 12}{4}$$

for $n \geq 4$ *and*

$$\sum_{i=1}^{n} cr_i^{neo} = \frac{5cr_n^{neo} + 5cr_{n-1}^{neo} - cr_{n-2}^{neo} - cr_{n-3}^{neo} - 8}{4}$$

for $n \geq 5$.

Proof. We note that $b_n^{\text{neo}} = 6b_{n-2}^{\text{neo}} - b_{n-4}^{\text{neo}} - 2$ for $n \geq 5$. Let n be odd, say $n = 2k - 1$ for some integer $k \geq 1$. Thus

$$b_5^{\text{neo}} = 6b_3^{\text{neo}} - b_1^{\text{neo}} - 2$$

$$b_6^{\text{neo}} = 6b_4^{\text{neo}} - b_2^{\text{neo}} - 2$$

$$b_7^{\text{neo}} = 6b_5^{\text{neo}} - b_3^{\text{neo}} - 2$$

$$\dots$$

$$b^{\text{neo}}_{2k-1} = 6b^{\text{neo}}_{2k-3} - b^{\text{neo}}_{2k-5} - 2$$

and hence

$$(b^{\text{neo}}_5 + b^{\text{neo}}_6 + \cdots + b^{\text{neo}}_{2k-1}) = 6(b^{\text{neo}}_3 + b^{\text{neo}}_4 + \cdots + b^{\text{neo}}_{2k-3}) - (b^{\text{neo}}_1 + b^{\text{neo}}_2 + \cdots + b^{\text{neo}}_{2k-5}) - 2(2k-5).$$

Thus

$$\begin{aligned}
&(b^{\text{neo}}_1 + b^{\text{neo}}_2 + \cdots + b^{\text{neo}}_{2k-1}) - (b^{\text{neo}}_1 + b^{\text{neo}}_2 + b^{\text{neo}}_3 + b^{\text{neo}}_4\ ) \\
&= 6(b^{\text{neo}}_1 + b^{\text{neo}}_2 + \cdots + b^{\text{neo}}_{2k-3} + b^{\text{neo}}_{2k-2} + b^{\text{neo}}_{2k-1}) \\
&\quad - 6(b^{\text{neo}}_1 + b^{\text{neo}}_2 + b^{\text{neo}}_{2k-2} + b^{\text{neo}}_{2k-1}) \\
&\quad - (b^{\text{neo}}_1 + b^{\text{neo}}_2 + \cdots 4b^{\text{neo}}_{2k-5} + b^{\text{neo}}_{2k-4} + b^{\text{neo}}_{2k-3} + b^{\text{neo}}_{2k-2} + b^{\text{neo}}_{2k-1}) \\
&\quad + b^{\text{neo}}_{2k-4} + b^{\text{neo}}_{2k-3} + b^{\text{neo}}_{2k-2} + b^{\text{neo}}_{2k-1} - 2(2k-5).
\end{aligned}$$

So

$$\begin{aligned}
\sum_{i=1}^{2k-1} b^{\text{neo}}_i - (b^{\text{neo}}_1 + b^{\text{neo}}_2 + b^{\text{neo}}_3 + b^{\text{neo}}_4) = 6 \sum_{i=1}^{2k-1} b^{\text{neo}}_i \ &- 6(b^{\text{neo}}_1 + b^{\text{neo}}_2 + b^{\text{neo}}_{2k-2} + b^{\text{neo}}_{2k-1}) \\
&- \sum_{i=1}^{2k-1} b^{\text{neo}}_i + (b^{\text{neo}}_{2k-4} + b^{\text{neo}}_{2k-3} + b^{\text{neo}}_{2k-2} + b^{\text{neo}}_{2k-1}) \\
&- 2(2k-5)
\end{aligned}$$

and hence

$$\sum_{i=1}^{2k-1} b^{\text{neo}}_i = \frac{5b^{\text{neo}}_{2k-1} + 5b^{\text{neo}}_{2k-2} - b^{\text{neo}}_{2k-3} - b^{\text{neo}}_{2k-4} + 2(2k-5)}{4}.$$

Since $n = 2k-1$, we conclude that

$$\sum_{i=1}^{n} b^{\text{neo}}_i = \frac{5b^{\text{neo}}_n + 5b^{\text{neo}}_{n-1} - b^{\text{neo}}_{n-2} - b^{\text{neo}}_{n-3} + 2n - 8}{4}.$$

The others can be proved similarly. □

We also note that

$$\sum_{i=1}^{n} (-1)^i B_i = \begin{cases} 2B^2_{\frac{n}{2}} + B_{\frac{n}{2}} C_{\frac{n}{2}} & n \geq 2 \text{ even} \\ -2B_{\frac{n+1}{2}}(b_{\frac{n+1}{2}} + \frac{1}{2}) & n \geq 1 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n} (-1)^i b_i = \begin{cases} 2B^2_{\frac{n}{2}} & n \geq 2 \text{ even} \\ -2b^2_{\frac{n+1}{2}} - 2b_{\frac{n+1}{2}} & n \geq 1 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i C_i = \begin{cases} B_n + 8B_{\frac{n}{2}}^2 & n \geq 2 \text{ even} \\ -B_n - 8(b_{\frac{n+1}{2}} + \frac{1}{2})^2 & n \geq 1 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i c_i = \begin{cases} B_n & n \geq 2 \text{ even} \\ -B_n & n \geq 1 \text{ odd.} \end{cases}$$

Similarly we can give the following theorem.

Theorem 4.2. *For the sums of neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers, we have*

$$\sum_{i=1}^{n}(-1)^i b_i^{neo} = \frac{1}{4}\begin{cases} 5b_n^{neo} - 5b_{n-1}^{neo} - b_{n-2}^{neo} + b_{n-3}^{neo} & n \geq 4 \text{ even} \\ -5b_n^{neo} + 5b_{n-1}^{neo} + b_{n-2}^{neo} - b_{n-3}^{neo} - 2 & n \geq 5 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i c_i^{neo} = \frac{1}{4}\begin{cases} 5c_n^{neo} - 5c_{n-1}^{neo} - c_{n-2}^{neo} + c_{n-3}^{neo} - 4 & n \geq 4 \text{ even} \\ -5c_n^{neo} + 5c_{n-1}^{neo} + c_{n-2}^{neo} - c_{n-3}^{neo} - 4 & n \geq 5 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i r_i^{neo} = \frac{1}{4}\begin{cases} 5r_n^{neo} - 5r_{n-1}^{neo} - r_{n-2}^{neo} + r_{n-3}^{neo} - 2 & n \geq 4 \text{ even} \\ -5r_n^{neo} + 5r_{n-1}^{neo} + r_{n-2}^{neo} - r_{n-3}^{neo} + 2 & n \geq 5 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i cr_i^{neo} = \frac{1}{4}\begin{cases} 5cr_n^{neo} - 5cr_{n-1}^{neo} - cr_{n-2}^{neo} + cr_{n-3}^{neo} - 4 & n \geq 6 \text{ even} \\ -5cr_n^{neo} + 5cr_{n-1}^{neo} + cr_{n-2}^{neo} - cr_{n-3}^{neo} - 4 & n \geq 5 \text{ odd.} \end{cases}$$

Proof. It can be proved as in the same way that Theorem 4.1 was proved. □

In [22, Lemma 1], Santana and Diaz-Barrero proved that the sum of first nonzero $4n+1$ terms of Pell numbers is a perfect square and is

$$\sum_{i=1}^{4n+1} P_i = \left(\sum_{i=0}^{n} \begin{pmatrix} 2n+1 \\ 2i \end{pmatrix} 2^i\right)^2.$$

Similarly we can give the following result.

Theorem 4.3. *The sum of first nonzero $4n+1$ terms of Pell numbers is*

$$\sum_{i=1}^{4n+1} P_i = \left(\frac{-b_{2n+3}^{neo} + 5b_{2n+2}^{neo} - 2}{3}\right)^2.$$

Proof. From Theorem 3.1, we notice that

$$-b_{2n+3}^{\text{neo}} + 5b_{2n+2}^{\text{neo}} = \frac{(-3+3\sqrt{2})\alpha^{2n+2} - (3+3\sqrt{2})\beta^{2n+2} + 4}{2}.$$

Since $\sum_{i=1}^{n} P_i = \frac{P_{n+1}+P_n-1}{2}$ and $P_n = \frac{\alpha^n-\beta^n}{2\sqrt{2}}$, we get

$$\begin{aligned}
\sum_{i=1}^{4n+1} P_i &= \frac{P_{4n+2}+P_{4n+1}-1}{2} \\
&= \frac{\frac{\alpha^{4n+2}-\beta^{4n+2}}{2\sqrt{2}} + \frac{\alpha^{4n+1}-\beta^{4n+1}}{2\sqrt{2}} - 1}{2} \\
&= \frac{\frac{\alpha^{4n+2}(1+\alpha^{-1})+\beta^{4n+2}(-1-\beta^{-1})}{2\sqrt{2}}}{2} - \frac{1}{2} \\
&= \frac{\alpha^{4n+2}+\beta^{4n+2}}{4} - \frac{1}{2} \\
&= \frac{(27-18\sqrt{2})\alpha^{4n+4} + (27+18\sqrt{2})\beta^{4n+4} - 18}{36} \\
&= \frac{1}{36}\left\{ \begin{array}{c} (-3+3\sqrt{2})^2\alpha^{4n+4} - 2(-3+3\sqrt{2})(3+3\sqrt{2})(\alpha\beta)^{2n+2} \\ +(3+3\sqrt{2})^2\beta^{4n+4} \end{array} \right\} \\
&= \left(\frac{(-3+3\sqrt{2})\alpha^{2n+2} - (3+3\sqrt{2})\beta^{2n+2}}{6} \right)^2 \\
&= \left(\frac{\frac{(-3+3\sqrt{2})\alpha^{2n+2}-(3+3\sqrt{2})\beta^{2n+2}+4}{2} - 2}{3} \right)^2 \\
&= \left(\frac{-b_{2n+3}^{\text{neo}} + 5b_{2n+2}^{\text{neo}} - 2}{3} \right)^2
\end{aligned}$$

as we wanted. □

As in Theorem 4.3, we can give the following theorem which can be proved similarly.

Theorem 4.4. *For the sums of Pell, Pell-Lucas, balancing and Lucas-cobalancing numbers, we have*

(1) *the sum of first nonzero* $4n-1$ *terms of Pell numbers* $+1$ *is a perfect square and is*

$$1 + \sum_{i=1}^{4n-1} P_i = (b_{2n+1}^{neo} - b_{2n}^{neo})^2.$$

(2) *the sum of odd ordered Pell-Lucas numbers from* 1 *to* $2n$ *is a perfect square and is*

$$\sum_{i=1}^{2n} Q_{2i-1} = \left(\frac{2b_{2n+1}^{neo} + 2b_{2n}^{neo} - 2}{3} \right)^2.$$

(3) *the half of the sum of odd ordered Pell-Lucas numbers from* 0 *to* $2n$ *is a perfect square and is*

$$\frac{\sum_{i=0}^{2n} Q_{2i+1}}{2} = \left(\frac{-b_{2n+3}^{neo} + 5b_{2n+2}^{neo} - 2}{3}\right)^2.$$

(4) *the sum of odd ordered balancing numbers from* 1 *to* $2n$ *is a perfect square and is*

$$\sum_{i=1}^{2n} B_{2i-1} = \left(\frac{b_{4n+1}^{neo} + b_{4n}^{neo} - 1}{6}\right)^2.$$

(5) *the sum of first* $4n+2$ *terms of Lucas-cobalancing numbers* $+1$ *is a perfect square and is*

$$1 + \sum_{i=1}^{4n+2} c_i = (b_{4n+3}^{neo} - b_{4n+2}^{neo})^2.$$

Panda and Ray proved in [15, Theorem 3.4] that the sum of first $2n-1$ Pell numbers is

$$\sum_{i=1}^{2n-1} P_i = B_n + b_n. \tag{4.1}$$

Later Gözeri, Özkoç and Tekcan proved in [7, Theorem 2.5] that the sum of Pell-Lucas numbers from 0 to $2n-1$ is

$$\sum_{i=0}^{2n-1} Q_i = C_n + c_n.$$

Since $R_n = b_n$, (4.1) becomes

$$\sum_{i=1}^{2n-1} P_i = B_n + R_n. \tag{4.2}$$

As in (4.2), we can give the following result.

Theorem 4.5. *The sum of first* $2n-1$ *Pell numbers is*

$$\sum_{i=1}^{2n-1} P_i = b_{2n-1}^{neo} + r_{2n}^{neo}.$$

Proof. We easily deduce that

$$\sum_{i=1}^{2n-1} P_i = \frac{P_{2n} + P_{2n-1} - 1}{2}$$

$$
\begin{aligned}
&= \frac{\frac{\alpha^{2n}-\beta^{2n}}{2\sqrt{2}} + \frac{\alpha^{2n-1}-\beta^{2n-1}}{2\sqrt{2}} - 1}{2} \\
&= \frac{\alpha^{2n}(1+\alpha^{-1}) + \beta^{2n}(-1-\beta^{-1})}{4\sqrt{2}} - \frac{1}{2} \\
&= \frac{\alpha^{2n}+\beta^{2n}}{4} - \frac{1}{2} \\
&= \frac{1}{4\sqrt{2}} \left\{ \begin{array}{c} \alpha^{2n}[(3+\sqrt{2})(-1+\sqrt{2})^2 - 5 + 4\sqrt{2}] \\ +\beta^{2n}[-(3-\sqrt{2})(-1-\sqrt{2})^2 + 5 + 4\sqrt{2}] \end{array} \right\} - \frac{1}{2} \\
&= \frac{1}{4\sqrt{2}} \left\{ \begin{array}{c} \alpha^{2n}[(3+\sqrt{2})\alpha^{-2} - 5 + 4\sqrt{2}] \\ +\beta^{2n}[-(3-\sqrt{2})\beta^{-2} + 5 + 4\sqrt{2}] \end{array} \right\} - \frac{1}{2} \\
&= \frac{1}{4\sqrt{2}} \left\{ \begin{array}{c} (3+\sqrt{2})\alpha^{2n-2} - (3-\sqrt{2})\beta^{2n-2} + 2\sqrt{2} \\ +(-5+4\sqrt{2})\alpha^{2n} + (5+4\sqrt{2})\beta^{2n} - 4\sqrt{2} \end{array} \right\} \\
&= b_{2n-1}^{\mathrm{neo}} + r_{2n}^{\mathrm{neo}}
\end{aligned}
$$

by Theorem 3.1. □

5. Neo Cobalancing Numbers and Balancing Numbers.

In Theorems 2.2, we deduced the general terms of all neo cobalancing numbers in terms of balancing and Lucas-balancing numbers and in Theorem 2.3, we deduced the general terms of them in terms of cobalancing and Lucas-cobalancing numbers. Conversely, we can give the general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers in terms of neo cobalancing numbers as follows:

Theorem 5.1. *The general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers are*

$$
\begin{aligned}
B_n &= \frac{b_{2n+1}^{neo} + b_{2n}^{neo} - 1}{6} \\
b_n &= \frac{b_{2n+1}^{neo} - 2b_{2n}^{neo} - 1}{3} \\
C_n &= b_{2n+1}^{neo} - b_{2n}^{neo} \\
c_n &= \frac{-b_{2n+1}^{neo} + 5b_{2n}^{neo} - 2}{3}
\end{aligned}
$$

for $n \geq 1$.

Proof. Recall that $B_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}$. So we get

$$\begin{aligned}
B_n &= \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}} \\
&= \frac{6(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}})}{6} \\
&= \frac{\frac{\alpha^{2n}(3+\sqrt{2}+3-\sqrt{2})+\beta^{2n}(-3+\sqrt{2}-3-\sqrt{2})\beta^{2n}+4\sqrt{2}}{4\sqrt{2}} - 1}{6} \\
&= \frac{\frac{(3+\sqrt{2})\alpha^{2n}-(3-\sqrt{2})\beta^{2n}+2\sqrt{2}}{4\sqrt{2}} + \frac{(3-\sqrt{2})\alpha^{2n}-(3+\sqrt{2})\beta^{2n}+2\sqrt{2}}{4\sqrt{2}} - 1}{6} \\
&= \frac{b^{\text{neo}}_{2n+1} + b^{\text{neo}}_{2n} - 1}{6}
\end{aligned}$$

by Theorem 3.1. The others can be proved similarly. □

6. Neo Cobalancing Numbers and Pell and Pell-Lucas Numbers.

Theorem 6.1. *The general terms of neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$\begin{aligned}
b^{neo}_{2n-1} &= \frac{4P_{2n-2} + P_{2n-3} + 1}{2} \\
b^{neo}_{2n} &= \frac{2P_{2n} - P_{2n-1} + 1}{2} \\
c^{neo}_{2n-1} &= 5P_{2n-2} + 3P_{2n-3} \\
c^{neo}_{2n} &= P_{2n} + 3P_{2n-1} \\
r^{neo}_{2n-1} &= \frac{P_{2n-2} + 2P_{2n-3} - 2}{2} \\
r^{neo}_{2n} &= \frac{-P_{2n} + 4P_{2n-1} - 2}{2} \\
cr^{neo}_{2n} &= 3P_{2n} - 5P_{2n-1}
\end{aligned}$$

for $n \geq 1$ *and*

$$cr^{neo}_{2n-1} = 3P_{2n-2} - P_{2n-3}$$

for $n \geq 2$.

Proof. From Theorem 3.1, we get

$$\begin{aligned} b_{2n}^{\text{neo}} &= \frac{(3-\sqrt{2})\alpha^{2n} - (3+\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}} \\ &= \frac{\alpha^{2n}(\frac{2-\alpha^{-1}}{2\sqrt{2}}) - \beta^{2n}(\frac{2-\beta^{-1}}{2\sqrt{2}}) + 2\sqrt{2}}{2} \\ &= \frac{2(\frac{\alpha^{2n}-\beta^{2n}}{2\sqrt{2}}) - (\frac{\alpha^{2n-1}-\beta^{2n-1}}{2\sqrt{2}}) + 1}{2} \\ &= \frac{2P_{2n} - P_{2n-1} + 1}{2}. \end{aligned}$$

The others can be proved similarly. □

Conversely, we can give the general terms of even and odd ordered Pell numbers in terms of neo cobalancing numbers and neo Lucas-cobalancing as follows:

Theorem 6.2. *The general terms of even and odd ordered Pell numbers are*

$$P_{2n} = \frac{6b_{2n}^{neo} + c_{2n}^{neo} - 3}{7} \quad \textit{and} \quad P_{2n-1} = \frac{2b_{2n-1}^{neo} + 2c_{2n-1}^{neo} - 1}{7}$$

for $n \geq 1$.

Proof. Applying Binet formulas, we deduce that

$$\begin{aligned} P_{2n} &= \frac{\alpha^{2n} - \beta^{2n}}{2\sqrt{2}} \\ &= \frac{\alpha^{2n}[\frac{6(3-\sqrt{2})}{4\sqrt{2}} + \frac{3-\sqrt{2}}{2}] + \beta^{2n}[\frac{-6(3+\sqrt{2})}{4\sqrt{2}} + \frac{3+\sqrt{2}}{2}]}{7} \\ &= \frac{6[\frac{(3-\sqrt{2})\alpha^{2n} - (3+\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}}] + [\frac{(3-\sqrt{2})\alpha^{2n} + (3+\sqrt{2})\beta^{2n}}{2}] - 3}{7} \\ &= \frac{6b_{2n}^{\text{neo}} + c_{2n}^{\text{neo}} - 3}{7}. \end{aligned}$$

The others can be proved similarly. □

Thus we construct one-to-one correspondence between all neo cobalancing numbers and Pell numbers.

Similarly, we can give the general terms of all neo cobalancing numbers in terms of Pell-Lucas numbers as follows:

Theorem 6.3. *The general terms of neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$b_{2n-1}^{neo} = \frac{5Q_{2n-2} + 3Q_{2n-3} + 4}{8}$$
$$b_{2n}^{neo} = \frac{Q_{2n} + 3Q_{2n-1} + 4}{8}$$
$$c_{2n-1}^{neo} = \frac{4Q_{2n-2} + Q_{2n-3}}{2}$$
$$c_{2n}^{neo} = \frac{2Q_{2n} - Q_{2n-1}}{2}$$
$$r_{2n-1}^{neo} = \frac{3Q_{2n-2} - Q_{2n-3} - 8}{8}$$
$$r_{2n}^{neo} = \frac{3Q_{2n} - 5Q_{2n-1} - 8}{8}$$
$$cr_{2n}^{neo} = \frac{-Q_{2n} + 4Q_{2n-1}}{2}$$

for $n \geq 1$ *and*

$$cr_{2n-1}^{neo} = \frac{Q_{2n-2} + 2Q_{2n-3}}{2}$$

for $n \geq 2$.

Proof. It can be proved as in the same way that Theorem 6.1 was proved. □

Conversely, we can give the general terms of even and odd ordered Pell-Lucas numbers in terms of all neo cobalancing numbers as follows.

Theorem 6.4. *The general terms of even and odd ordered Pell-Lucas numbers are*

$$Q_{2n} = \frac{8c_{2n}^{neo} + 2cr_{2n}^{neo}}{7} \quad \text{and} \quad Q_{2n-1} = \frac{20b_{2n-1}^{neo} + 4r_{2n-1}^{neo} - 6}{7}$$

for $n \geq 1$.

Proof. It can be proved as in the same way that Theorem 6.2 was proved. □

Thus we construct one-to-one correspondence between all neo cobalancing numbers and Pell-Lucas numbers.

7. Neo Cobalancing Numbers and Triangular and Square Triangular Numbers.

Triangular numbers denoted by T_n are the numbers of the form

$$T_n = \frac{n(n+1)}{2}.$$

It is known that there is a correspondence between balancing and also cobalancing numbers and triangular numbers. From (1.1), we get

$$\frac{(n+r)(n+r+1)}{2} = n^2.$$

So

$$T_{B_n+R_n} = B_n^2.$$

Similarly from (1.3), we get

$$\frac{(n+r)(n+r+1)}{2} = n^2 + n.$$

So

$$T_{b_n+r_n} = b_n^2 + b_n. \tag{7.1}$$

As in (7.1), we can give the following theorem.

Theorem 7.1. *b_n^{neo} is a neo cobalancing number if and only if $(b_n^{neo})^2 - b_n^{neo} + 1$ is a triangular number, that is,*

$$T_{b_n^{neo}+r_n^{neo}} = (b_n^{neo})^2 - b_n^{neo} + 1.$$

Proof. From (2.2), we get

$$n^2 + n = 4n - 2 + 2nr + r^2 + r$$

and hence

$$\frac{(n+r)(n+r+1)}{2} = n^2 - n + 1.$$

Thus

$$T_{b_n^{\text{neo}}+r_n^{\text{neo}}} = (b_n^{\text{neo}})^2 - b_n^{\text{neo}} + 1$$

as we wanted. □

There are infinitely many triangular numbers that are also square numbers which are called square triangular numbers and is denoted by S_n. So

$$S_n = s_n^2 = \frac{t_n(t_n+1)}{2},$$

where s_n and t_n are the sides of the corresponding square and triangle. Their Binet formulas are

$$S_n = \frac{\alpha^{4n}+\beta^{4n}-2}{32},\ s_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}} \text{ and } t_n = \frac{\alpha^{2n}+\beta^{2n}-2}{4} \tag{7.2}$$

for $n \geq 1$.

We can give the general terms of all neo cobalancing numbers in terms of s_n and t_n as follows.

Theorem 7.2. *The general terms of neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$\begin{aligned} b_{2n-1}^{neo} &= 3s_{n-1} + t_{n-1} + 1 \\ b_{2n}^{neo} &= 3s_n - t_n \\ c_{2n-1}^{neo} &= 4s_{n-1} + 6t_{n-1} + 3 \\ c_{2n}^{neo} &= -4s_n + 6t_n + 3 \\ r_{2n-1}^{neo} &= -s_{n-1} + 2t_{n-1} \\ r_{2n}^{neo} &= -5s_n + 4t_n + 1 \\ cr_{2n}^{neo} &= 16s_n - 10t_n - 5 \end{aligned}$$

for $n \geq 1$ and

$$cr_{2n-1}^{neo} = 8s_{n-1} - 2t_{n-1} - 1$$

for $n \geq 2$.

Proof. From Theorem 3.1, we get

$$\begin{aligned} b_{2n-1}^{\text{neo}} &= \frac{(3+\sqrt{2})\alpha^{2n-2} - (3-\sqrt{2})\beta^{2n-2} + 2\sqrt{2}}{4\sqrt{2}} \\ &= \alpha^{2n-2}(\frac{3}{4\sqrt{2}} + \frac{1}{4}) - \beta^{2n-2}(\frac{3}{4\sqrt{2}} - \frac{1}{4}) - \frac{1}{2} + 1 \\ &= 3(\frac{\alpha^{2n-2}-\beta^{2n-2}}{4\sqrt{2}}) + (\frac{\alpha^{2n-2}+\beta^{2n-2}-2}{4}) + 1 \end{aligned}$$

$$= 3s_{n-1} + t_{n-1} + 1.$$

by (7.2). The others can be proved similarly. □

Conversely, we can give the following theorem.

Theorem 7.3. *The general terms of S_n, s_n and t_n are*

$$S_n = \frac{(b_{2n+1}^{neo})^2 + (b_{2n}^{neo})^2 + 2b_{2n+1}^{neo}b_{2n}^{neo} - 2b_{2n+1}^{neo} - 2b_{2n}^{neo} + 1}{36}$$
$$s_n = \frac{b_{2n+1}^{neo} + b_{2n}^{neo} - 1}{6}$$
$$t_n = \frac{b_{2n+1}^{neo} - b_{2n}^{neo} - 1}{2}$$

for $n \geq 1$.

Proof. From (7.2), we get

$$\begin{aligned}
S_n &= \frac{\alpha^{4n} + \beta^{4n} - 2}{32} \\
&= \frac{1}{36}\left(\frac{9}{8}(\alpha^{4n} + \beta^{4n}) - \frac{9}{4}(\alpha\beta)^{2n}\right) \\
&= \frac{1}{36}\left\{\begin{array}{c}
\left(\begin{array}{c} \frac{1}{4} + \frac{3\sqrt{2}}{8}(\alpha^{2n} - \beta^{2n}) + \frac{11}{32}(\alpha^{4n} + \beta^{4n}) + \frac{3\sqrt{2}}{16}(\alpha^{4n} - \beta^{4n}) \\ +\frac{1}{4}(\alpha^{2n} + \beta^{2n}) - \frac{7}{16}(\alpha\beta)^{2n} \end{array}\right) \\
+\left(\begin{array}{c} \frac{1}{4} + \frac{3\sqrt{2}}{8}(\alpha^{2n} - \beta^{2n}) + \frac{11}{32}(\alpha^{4n} + \beta^{4n}) - \frac{3\sqrt{2}}{16}(\alpha^{4n} - \beta^{4n}) \\ -\frac{1}{4}(\alpha^{2n} + \beta^{2n}) - \frac{7}{16}(\alpha\beta)^{2n} \end{array}\right) \\
+\left(\frac{1}{2} + \frac{3\sqrt{2}}{4}(\alpha^{2n} - \beta^{2n}) + \frac{7}{16}(\alpha^{4n} + \beta^{4n}) - \frac{11}{8}(\alpha\beta)^{2n}\right) \\
+\left(-\frac{3\sqrt{2}}{2}(\alpha^{2n} - \beta^{2n}) - 1\right)
\end{array}\right\} \\
&= \frac{1}{36}\left\{\begin{array}{c}
\left(\frac{(3+\sqrt{2})\alpha^{2n} - (3-\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}}\right)^2 + \left(\frac{(3-\sqrt{2})\alpha^{2n} - (3+\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}}\right)^2 \\
+2\left(\frac{(3+\sqrt{2})\alpha^{2n} - (3-\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}}\right)\left(\frac{(3-\sqrt{2})\alpha^{2n} - (3+\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}}\right) \\
-2\left(\frac{(3+\sqrt{2})\alpha^{2n} - (3-\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}}\right) - 2\left(\frac{(3-\sqrt{2})\alpha^{2n} - (3+\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}}\right) + 1
\end{array}\right\} \\
&= \frac{(b_{2n+1}^{\mathrm{neo}})^2 + (b_{2n}^{\mathrm{neo}})^2 + 2b_{2n+1}^{\mathrm{neo}}b_{2n}^{\mathrm{neo}} - 2b_{2n+1}^{\mathrm{neo}} - 2b_{2n}^{\mathrm{neo}} + 1}{36}
\end{aligned}$$

by Theorem 3.1. The others can be proved similarly. □

Thus we construct one-to-one correspondence between all neo cobalancing numbers and square triangular numbers.

Finally, we want to construct a correspondence between triangular and square triangular numbers via neo cobalancing numbers, that is, we want to find out that for which neo cobalancing numbers (or neo cobalancers) m, the equation

$$T_m = S_n$$

holds. The answer is given below.

Theorem 7.4. *For triangular numbers T_n and square triangular numbers S_n, we have*

(1) $T_{\frac{b^{neo}_{2n+1}-b^{neo}_{2n}-1}{2}} = S_n$
(2) $T_{\frac{b^{neo}_{2n+1}+3r^{neo}_{2n+1}-1}{7}} = S_n$
(3) $T_{\frac{5r^{neo}_{2n+1}-r^{neo}_{2n}+1}{6}} = S_n$

for $n \geq 1$.

Proof. **(1)** From Theorem 3.1, we note that $b^{\text{neo}}_{2n+1} - b^{\text{neo}}_{2n} = \frac{\alpha^{2n}+\beta^{2n}}{2}$. So we get

$$\begin{aligned}
T_{\frac{b^{\text{neo}}_{2n+1}-b^{\text{neo}}_{2n}-1}{2}} &= \frac{(\frac{b^{\text{neo}}_{2n+1}-b^{\text{neo}}_{2n}-1}{2})(\frac{b^{\text{neo}}_{2n+1}-b^{\text{neo}}_{2n}-1}{2}+1)}{2} \\
&= \frac{(b^{\text{neo}}_{2n+1}-b^{\text{neo}}_{2n}-1)(b^{\text{neo}}_{2n+1}-b^{\text{neo}}_{2n}+1)}{8} \\
&= \frac{(\frac{\alpha^{2n}+\beta^{2n}}{2}-1)(\frac{\alpha^{2n}+\beta^{2n}}{2}+1)}{8} \\
&= \frac{(\frac{\alpha^{2n}+\beta^{2n}}{2})^2-1}{8} \\
&= \frac{\alpha^{4n}+\beta^{4n}-2}{32} \\
&= S_n
\end{aligned}$$

by (7.2). The others can be proved similarly. □

8. Cassini and Catalan Identities.

Note that Cassini identity for Fibonacci numbers F_n is

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

for $n \geq 1$ and Catalan identity for Fibonacci numbers F_n is

$$F_n^2 - F_{n-m}F_{n+m} = (-1)^{n-m}F_m^2$$

for $n \geq m \geq 1$. Similarly for all neo cobalancing numbers, we can give the following theorems.

Theorem 8.1. *Cassini identities for all neo cobalancing numbers are*

$$\begin{aligned}
b_{2n}^{neo}b_{2n+2}^{neo} - (b_{2n+1}^{neo})^2 &= -2B_nb_{n+1} - 1\\
b_{2n-1}^{neo}b_{2n+1}^{neo} - (b_{2n}^{neo})^2 &= 2B_n(b_n+1) - 1\\
c_{2n}^{neo}c_{2n+2}^{neo} - (c_{2n+1}^{neo})^2 &= -c_{2n+1} + 7\\
c_{2n-1}^{neo}c_{2n+1}^{neo} - (c_{2n}^{neo})^2 &= c_{2n} + 7\\
r_{2n}^{neo}r_{2n+2}^{neo} - (r_{2n+1}^{neo})^2 &= \frac{(c_n+2)(2-C_n)-1}{4}\\
r_{2n-1}^{neo}r_{2n+1}^{neo} - (r_{2n}^{neo})^2 &= \frac{(c_n-2)(C_{n-1}-2)-1}{4}\\
cr_{2n}^{neo}cr_{2n+2}^{neo} - (cr_{2n+1}^{neo})^2 &= -c_{2n} - 7
\end{aligned}$$

for $n \geq 1$ *and*

$$cr_{2n-1}^{neo}cr_{2n+1}^{neo} - (cr_{2n}^{neo})^2 = c_{2n-1} - 7$$

for $n \geq 2$.

Proof. Applying Theorem 3.1, we deduce that

$$\begin{aligned}
b_{2n}^{\text{neo}}b_{2n+2}^{\text{neo}} - (b_{2n+1}^{\text{neo}})^2 &= \left(\frac{(3-\sqrt{2})\alpha^{2n} - (3+\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}}\right)\\
&\quad\times\left(\frac{(3-\sqrt{2})\alpha^{2n+2} - (3+\sqrt{2})\beta^{2n+2} + 2\sqrt{2}}{4\sqrt{2}}\right)\\
&\quad-\left(\frac{(3+\sqrt{2})\alpha^{2n} - (3-\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}}\right)^2\\
&= \frac{1}{32}\left\{\begin{array}{c} 4\sqrt{2}(\alpha^{2n} - \beta^{2n}) - 2\sqrt{2}(\alpha^{4n} - \beta^{4n})\\ -2(\alpha^{4n} + \beta^{4n}) - 28 \end{array}\right\}\\
&= \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} + \frac{\alpha^{4n}(-2-2\sqrt{2}) + \beta^{4n}(-2+2\sqrt{2})}{32} - \frac{7}{8}
\end{aligned}$$

$$= \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} - \frac{\alpha^{4n+1} + \beta^{4n+1}}{16} - \frac{7}{8}$$
$$= -2\left(\frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}\right)\left(\frac{\alpha^{2n+1} - \beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2}\right) - 1$$
$$= -2B_n b_{n+1} - 1.$$

The other cases can be proved similarly. □

Apart from Theorem 8.1, we can give the Cassini identities for all neo cobalancing numbers in terms of neo cobalancing numbers as follows:

Theorem 8.2. *Cassini identities for all neo cobalancing numbers are*

$$b_{2n}^{neo} b_{2n+2}^{neo} - (b_{2n+1}^{neo})^2 = -1 - \frac{(b_{2n+1}^{neo} + b_{2n}^{neo} - 1)(b_{2n+3}^{neo} - 2b_{2n+2}^{neo} - 1)}{9}$$
$$b_{2n-1}^{neo} b_{2n+1}^{neo} - (b_{2n}^{neo})^2 = -1 + \frac{(b_{2n+1}^{neo} + b_{2n}^{neo} - 1)(b_{2n+1}^{neo} - 2b_{2n}^{neo} + 2)}{9}$$
$$c_{2n}^{neo} c_{2n+2}^{neo} - (c_{2n+1}^{neo})^2 = 7 - \frac{c_{4n+2}^{neo} - c_{4n+1}^{neo}}{2}$$
$$c_{2n-1}^{neo} c_{2n+1}^{neo} - (c_{2n}^{neo})^2 = 7 + \frac{c_{4n}^{neo} - c_{4n-1}^{neo}}{2}$$
$$r_{2n}^{neo} r_{2n+2}^{neo} - (r_{2n+1}^{neo})^2 = \frac{5}{24} - \frac{r_{2n+1}^{neo} + r_{2n}^{neo}}{3} - \frac{r_{4n+1}^{neo} - r_{4n}^{neo}}{8}$$
$$r_{2n-1}^{neo} r_{2n+1}^{neo} - (r_{2n}^{neo})^2 = \frac{-1}{8} + \frac{5r_{4n+1}^{neo} - 13r_{4n}^{neo}}{24} - \frac{r_{2n+1}^{neo} + r_{2n}^{neo}}{3}$$
$$cr_{2n}^{neo} cr_{2n+2}^{neo} - (cr_{2n+1}^{neo})^2 = -7 - \frac{cr_{4n+1}^{neo} + cr_{4n}^{neo}}{6}$$

for $n \geq 1$ *and*

$$cr_{2n-1}^{neo} cr_{2n+1}^{neo} - (cr_{2n}^{neo})^2 = -7 + \frac{3cr_{4n}^{neo} - cr_{4n+1}^{neo}}{2}$$

for $n \geq 2$.

Proof. It can be proved as in the same way that Theorem 8.1 was proved. □

Finally as in Theorem 8.2, we can give the following result concerning Catalan identities for all neo cobalancing numbers.

Theorem 8.3. *Catalan identities for neo cobalancing numbers are*

(1) $(b_{2n}^{neo})^2 - b_{2n-2m}^{neo} b_{2n+2m}^{neo} = b_{2n}^{neo} - (b_{2m+1}^{neo} - b_{2m}^{neo})(b_{2n}^{neo} - \frac{1}{2}) + (7b_{4m+1}^{neo} - 7b_{4m}^{neo} - 15)/16$ *for* $n > m \geq 1$.

(2) $(b_{2n-1}^{neo})^2 - b_{2n-2m-1}^{neo} b_{2n+2m-1}^{neo} = b_{2n-1}^{neo} - (b_{2m+1}^{neo} - b_{2m}^{neo})(b_{2n-1}^{neo} - \frac{1}{2}) + (7b_{4m+1}^{neo} - 7b_{4m}^{neo} - 15)/16$ *for* $n > m \geq 1$.

(3) $(b_{2n}^{neo})^2 - b_{2n-2m+1}^{neo} b_{2n+2m-1}^{neo} = b_{2n}^{neo} + (b_{4n+1}^{neo} - 5b_{4n}^{neo})/24 - (b_{2n-2m+1}^{neo} + b_{2n+2m-1}^{neo})/2 + (21b_{4m-1}^{neo} - 21b_{4m-2}^{neo} - 17)/48$ *for* $n \geq m \geq 1$.

(4) $(b_{2n-1}^{neo})^2 - b_{2n-2m}^{neo} b_{2n+2m-2}^{neo} = b_{2n-1}^{neo} - (b_{2n-2m}^{neo} + b_{2n+2m-2}^{neo})/2 + (5b_{4n-2}^{neo} - b_{4n-1}^{neo})/24 + (21b_{4m-1}^{neo} - 21b_{4m-2}^{neo} - 25)/48$ *for* $n > m \geq 1$.

for neo Lucas-cobalancing numbers are

(1) $(c_{2n}^{neo})^2 - c_{2n-2m}^{neo} c_{2n+2m}^{neo} = (-21c_{4m}^{neo} + 7c_{4m-1}^{neo} + 28)/8$ *for* $n > m \geq 1$.

(2) $(c_{2n-1}^{neo})^2 - c_{2n-2m-1}^{neo} c_{2n+2m-1}^{neo} = (-21c_{4m}^{neo} + 7c_{4m-1}^{neo} + 28)/8$ *for* $n > m \geq 1$.

(3) $(c_{2n}^{neo})^2 - c_{2n-2m+1}^{neo} c_{2n+2m-1}^{neo} = (c_{4n-1}^{neo} - c_{4n}^{neo} + 7)/2 - (7c_{4m-1}^{neo} + 7c_{4m-2}^{neo} - 42)/12$ *for* $n \geq m \geq 1$.

(4) $(c_{2n-1}^{neo})^2 - c_{2n-2m}^{neo} c_{2n+2m-2}^{neo} = c_{4n}^{neo} - 2c_{4n-1}^{neo} - (7c_{4m-1}^{neo} + 7c_{4m-2}^{neo} - 42)/12$ *for* $n > m \geq 1$.

for neo cobalancers are

(1) $(r_{2n}^{neo})^2 - r_{2n-2m}^{neo} r_{2n+2m}^{neo} = -2r_{2n}^{neo} + (r_{2n}^{neo} + 1)(10r_{2m+1}^{neo} - 2r_{2m}^{neo} + 8)/3 - (35r_{4m+1}^{neo} - 7r_{4m}^{neo} + 103)/48$ *for* $n > m \geq 1$.

(2) $(r_{2n-1}^{neo})^2 - r_{2n-2m-1}^{neo} r_{2n+2m-1}^{neo} = -2r_{2n-1}^{neo} + (r_{2n-1}^{neo} + 1)(10r_{2m+1}^{neo} - 2r_{2m}^{neo} + 8)/3 - (35r_{4m+1}^{neo} - 7r_{4m}^{neo} + 103)/48$ *for* $n > m \geq 1$.

(3) $(r_{2n}^{neo})^2 - r_{2n-2m+1}^{neo} r_{2n+2m-1}^{neo} = (-10r_{4n+1}^{neo} + 26r_{4n}^{neo} - 35r_{4m-1}^{neo} + 7r_{4m-2}^{neo} + 9)/48 - 2r_{2n}^{neo} + r_{2n-2m+1}^{neo} + r_{2n+2m-1}^{neo}$ *for* $n \geq m \geq 1$.

(4) $(r_{2n-1}^{neo})^2 - r_{2n-2m}^{neo} r_{2n+2m-2}^{neo} = -2r_{2n-1}^{neo} + r_{2n-2m}^{neo} + r_{2n+2m-2}^{neo} + (r_{4n-3}^{neo} - r_{4n-4}^{neo})/8 - (35r_{4m-1}^{neo} - 7r_{4m-2}^{neo} + 7)/48$ *for* $n > m \geq 1$.

and for neo Lucas-cobalancers are

(1) $(cr_{2n}^{neo})^2 - cr_{2n-2m}^{neo} cr_{2n+2m}^{neo} = (14cr_{4m+1}^{neo} - 7cr_{4m}^{neo} - 21)/6$ *for* $n > m \geq 1$.

(2) $(cr_{2n-1}^{neo})^2 - cr_{2n-2m-1}^{neo} cr_{2n+2m-1}^{neo} = (14cr_{4m+1}^{neo} - 7cr_{4m}^{neo} - 21)/6$ *for* $n > m + 1 \geq 0$.

(3) $(cr_{2n}^{neo})^2 - cr_{2n-2m+1}^{neo} cr_{2n+2m-1}^{neo} = (-3cr_{4n}^{neo} + cr_{4n+1}^{neo})/2 + (7cr_{4m}^{neo} - 7cr_{4m-1}^{neo} - 14)/4$ *for* $n > m \geq 1$.

(4) $(cr_{2n-1}^{neo})^2 - cr_{2n-2m}^{neo} cr_{2n+2m-2}^{neo} = (-9cr_{4n}^{neo} + 19cr_{4n-1}^{neo})/4 + (7cr_{4m}^{neo} - 7cr_{4m-1}^{neo} - 14)/4$ *for* $n > m \geq 1$.

Proof. It can be proved as in the same way that Theorem 8.1 was proved. □

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