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Propagation of dark-bright soliton and kink wave solutions of fluidized granular matter model arising in industrial applications

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Abstract: The improved $\tan(\varphi/2)$-expansion, simplest equation, and extended (G′/G)-expansion methods are employed to construct the exact solutions involving parameters of the Van der Waals equation arising in the material industry. This model explains the phase separation phenomenon. Understanding the prominent dynamic and static properties of this model and other models of this type is of great importance for the physical phenomena encountered in many areas of industry. Therefore, for such models, it is also important to obtain guiding solutions in obtaining new information. Many explicit wave solutions consisting of trigonometric, hyperbolic, rational, and exponential functions are found by using analytical techniques. The obtained solutions were verified with Maple by placing them back into the original equations. Moreover, graphical demonstrations for some of the obtained solutions are given.

Keywords: extended (G'/G)-expansion method; ITEM; simplest equation method; van der Waals equation.

MSC 2010: 35C07; 35C08.

Nomenclature

u	the flow velocity
η	the bulk (effective) viscosity
ϵ	the bifurcation parameter
k	the arbitrary wave constant
l	the wave velocity
C_1, C_2	the arbitrary constants of integration

1 Introduction

Nonlinear phenomena play an important role in applied mathematics and physics as a result of modeling real-world problems. Calculation of exact solutions or obtaining an approximation to these solutions, especially traveling wave solutions of nonlinear evolution equations (NLEEs) in mathematical physics, have an important place in soliton theory. Advances in computer science have made it more accessible to obtain exact solutions to these equations using symbolic computer programs such as Maple, Matlab, and Mathematica, which facilitate complex and tedious algebraic calculations.

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NLEEs appear in various fields such as engineering, optical fibers, solid mechanics, chemistry, biology, acoustic waves in crystals, biophysics, mechanics, quantum field theory and applied physics, and in various models that exist in nature. Numerous methods have been developed by researchers to reach exact solutions. Some of these well-known methods include: the generalised modified Tanh–Coth method [1], extended sinh-Gordon equation expansion method [2–4], the modified $\exp(-\phi(\xi))$-expansion function method [5], the (G'/G)-expansion method [6, 7], the homogeneous balance method [8], the inverse scattering transform method [9], the Hirota's bilinear method [10, 11], improved $\tan(\varphi/2)$–expansion method [12–16], the unified method [17], the modified simple equation method [18], the generalized Kudryashov method [19, 20], the extended modified auxiliary equation mapping method [21], the generalized direct algebraic method [22], the extended trial equation method [23], the hyperbolic function method [24, 25], the sub-ordinary differential equations method [26], the auxiliary equation mapping method [27] and so on [28–37]. These are some of the wide variety of direct and powerful techniques developed to determine numerical and analytical solutions to NLEEs. These solutions are important for making sense of physical phenomena in various fields, obtaining more detailed information and moving forward, and have attracted a lot of attention in recent years. The contributions of scientists to the literature, both in terms of introducing new methods and successfully applying existing methods, continue.

In this paper, we study one of the such NLEEs namely the Van der Waals form as follows [38, 39]

$$\frac{\partial^2}{\partial t^2}u + \frac{\partial^4}{\partial x^4}u - \eta\,\frac{\partial^3}{\partial x^2\partial t}u - 6\,u\left(\frac{\partial}{\partial x}u\right)^2 - 3\,u^2\frac{\partial^2}{\partial x^2}u - \epsilon\,\frac{\partial^2}{\partial x^2}u = 0, \tag{1}$$

where $u(x,t)$ describes the field which reflect correction to critical average vertical density, x is the horizontal direction of the granular system. Here, ϵ is the bifurcation parameter, η is effective viscosity, respectively. This equation models fluidized granular matter and the phase separation phenomenon. The granular material, which is fluidized by continuous energy injection, distributes the kinetic energy of the particles to the internal degrees of freedom of the particles in collisions. Therefore, energy must be supplied continuously to maintain a fluid regime. Experimentally, energy is often injected through vibrating walls or a gravitational field. Fluizied granular material regimes have been successfully described using hydrodynamic models [40, 42]. The Eq. (1), which we encounter with its applications in geophysics, civil engineering, pharmaceuticals, and industrial applications, has a very important role. Larger particle forms are obtained from the primary powder particles by the granulation process. For instance, friction is an example of granulation known as the result of particle collision. The two types of granular matter are granular solids and gases. Granular solids describe a substance in which the particles are fairly stationary when the average energy of each particle is less, while granular gas defines the substance in which contact between particles is very rare. Some examples of granular material are rice, sand, fertilizer, etc. In addition, powders are known to be special granular materials because they are a mixture of micro-sized particles. This model, which has an important place in many physical phenomena and especially in the material industry, has been the subject of many studies such as [38, 39, 41]. Three different methods were used to solve the van der Waals normal form analytically in [42]. In [38], exponential rational function and generalized Kudryashov methods were applied to seek new exact solutions of Eq. (1). In [39], six different methods for this model were employed to obtain the traveling and solitary wave solutions. In the study published in 2020 [41], the hyperbolic function method and the modified Kudryashov method were used in the model under consideration. In this regard, in order to contribute to the literature, we will apply three effective methods that are not applied for the relevant model to find new solutions. These new constructed solutions play vital role in the field of material industry, where this model arises.

The paper is regulated as follows: In Section 2, we give the basic structures of the improved $\tan(\varphi/2)$-expansion method (ITEM), the simplest equation method (SEM), and the extended G'/G-expansion method, respectively. The governing equation is obtained in Section 3. In Sections 4–6, we implement these methods to obtain the exact traveling and solitary wave solutions of the underlying equation. In Section 7, conclusions are given.

2 Description of methods

In this section, we will go over the main points of the ITEM, the simplest equation and the extended G'/G-expansion methods to construct various exact solitary wave solutions.

2.1 The ITEM for NLEEs

In this section, we will give the basic principles of the ITEM, which was developed and applied by Manafian et al. in 2015 [12] as a useful method for finding analytical solutions. Consider an NLEE given in the form below:

$$\Xi(q, q_x, q_t, q_{xx}, q_{tt}, \ldots) = 0. \tag{2}$$

In (2), using traveling wave transformation $\xi = kx + vt, q(x,t) = q(\xi)$ the following ordinary differential equation (ODE) is obtained:

$$\Omega(q, kq', vq', k^2q'', v^2q'', \ldots) = 0. \tag{3}$$

Suppose the solutions of this reduced Eq. (3) take the form

$$q(\xi) = S(\varphi) = \sum_{s=-j}^{j} A_s[\varrho + \tan(\varphi/2)]^s \tag{4}$$

where $A_s(0 \le s \le j)$ and $A_{-s} = B_s(1 \le s \le j)(A_j \ne 0, B_j \ne 0)$ and ϱ are arbitrary constants. Here, $\varphi = \varphi(\xi)$ satisfies the first order ODE given below:

$$\varphi'(\xi) = \alpha\ \sin(\varphi(\xi)) + \beta\ \cos(\varphi(\xi)) + c \tag{5}$$

where α, β and c are arbitrary constants. The solution families that differ according to the relations between the coefficients of the ODE (5) are given below.

Family 1. When $\Delta = \alpha^2 + \beta^2 - c^2 < 0$ and $\beta - c \ne 0$, then $\varphi(\xi) = 2\tan^{-1}\left[\frac{\alpha}{\beta-c} - \frac{\sqrt{-\Delta}}{\beta-c}\ \tan(\frac{\sqrt{-\Delta}}{2}\bar{\xi})\right]$.

Family 2. When $\Delta = \alpha^2 + \beta^2 - c^2 > 0$ and $\beta - c \ne 0$, then $\varphi(\xi) = 2\tan^{-1}\left[\frac{\alpha}{\beta-c} + \frac{\sqrt{\Delta}}{\beta-c}\ \tanh(\frac{\sqrt{\Delta}}{2}\bar{\xi})\right]$.

Family 3. When $\Delta = \alpha^2 + \beta^2 - c^2 > 0$, $\beta \ne 0$, and $c = 0$, then $\varphi(\xi) = 2\tan^{-1}\left[\frac{\alpha}{\beta} + \frac{\sqrt{\beta^2+\alpha^2}}{\beta}\ \tanh(\frac{\sqrt{\beta^2+\alpha^2}}{2}\bar{\xi})\right]$.

Family 4. When $\Delta = \alpha^2 + \beta^2 - c^2 < 0, c \ne 0$, and $\beta = 0$, then $\varphi(\xi) = 2\tan^{-1}\left[\frac{-\alpha}{c} + \frac{\sqrt{c^2-\alpha^2}}{c}\ \tan(\frac{\sqrt{c^2-\alpha^2}}{2}\bar{\xi})\right]$.

Family 5. When $\Delta = \alpha^2 + \beta^2 - c^2 > 0, \beta - c \ne 0$, and $\alpha = 0$, then $\varphi(\xi) = 2\tan^{-1}\left[\sqrt{\frac{\beta+c}{\beta-c}}\ \tanh(\frac{\sqrt{\beta^2-c^2}}{2}\bar{\xi})\right]$.

Family 6. When $\alpha = 0$ and $c = 0$, then $\varphi(\xi) = \tan^{-1}\left[\frac{e^{2\beta\bar{\xi}}-1}{e^{2\beta\bar{\xi}}+1}, \frac{e^{2\beta\bar{\xi}}}{e^{2\beta\bar{\xi}}+1}\right]$.

Family 7. When $\beta = 0$ and $c = 0$, then $\varphi(\xi) = \tan^{-1}\left[\frac{e^{2\alpha\bar{\xi}}}{e^{2\alpha\bar{\xi}}+1}, \frac{e^{2\alpha\bar{\xi}}-1}{e^{2\alpha\bar{\xi}}+1}\right]$.

Family 8. When $\alpha^2 + \beta^2 = c^2$, then $\varphi(\xi) = 2\tan^{-1}\left[\frac{\alpha\bar{\xi}+2}{(\beta-c)\bar{\xi}}\right]$.

Family 9. When $\alpha = \beta = c = k\alpha$, then $\varphi(\xi) = 2\tan^{-1}\left[e^{k\alpha\bar{\xi}} - 1\right]$.

Family 10. When $\alpha = c = k\alpha$ and $\beta = -k\alpha$, then $\varphi(\xi) = -2\tan^{-1}\left[\frac{e^{k\alpha\bar{\xi}}}{e^{k\alpha\bar{\xi}}-1}\right]$.

Family 11. When $c = \alpha$, then $\varphi(\xi) = -2\tan^{-1}\left[\frac{(\alpha+\beta)e^{\beta\bar{\xi}}-1}{(\alpha-\beta)e^{\beta\bar{\xi}}-1}\right]$.

Family 12. When $\alpha = c$, then $\varphi(\xi) = 2\tan^{-1}\left[\frac{(c+\beta)e^{\beta\bar{\xi}}+1}{(\beta-c)e^{\beta\bar{\xi}}-1}\right]$.

Family 13. When $c = -\alpha$, then $\varphi(\xi) = 2\tan^{-1}\left[\frac{e^{\beta\bar{\xi}}+\beta-\alpha}{e^{\beta\bar{\xi}}-\beta-\alpha}\right]$.

Family 14. When $\beta = -c$, then $\varphi(\xi) = 2\tan^{-1}\left[\frac{\alpha e^{\alpha\bar{\xi}}}{1-ce^{\alpha\bar{\xi}}}\right]$.

Family 15. When $\beta = 0$ and $\alpha = c$, then $\varphi(\xi) = -2\tan^{-1}\left[\frac{c\bar{\xi}+2}{c\bar{\xi}}\right]$.

Family 16. When $\alpha = 0$ and $\beta = c$, then $\varphi(\xi) = 2\tan^{-1}\left[c\bar{\xi}\right]$.

Family 17. When $\alpha = 0$ and $\beta = -c$, then $\varphi(\xi) = -2\tan^{-1}\left[\frac{1}{c\bar{\xi}}\right]$.

Family 18. When $\alpha = 0$ and $\beta = 0$, then $\varphi(\xi) = c\xi + D$.

Family 19. When $\beta = c$, then $\varphi(\xi) = 2\tan^{-1}\left[\frac{e^{\alpha\bar{\xi}}-c}{\alpha}\right]$, where $\bar{\xi} = \xi + C$, p, A_0, A_s, and $B_s (s = 1, 2, \ldots, m)$ are constants.

As is often done in similar methods, balancing the highest order derivatives with the highest order nonlinear terms in Eq. (3), one can acquire j. Following determining j, if Eq. (4) is substituted into reduced Eq. (3), an algebraic equations set which contains $\tan(\varphi/2)^s$, $\cot(\varphi/2)^s$, $(s = 0, 1, 2, \ldots)$ is obtained. Then setting each coefficients of $\tan(\varphi/2)^s$, $\cot(\varphi/2)^s$ to zero, we can get a set of over-determined equations for $A_0, A_s, B_s (s = 1, 2, \ldots, j), \alpha, \beta, c$ and ϱ. Using computer programming such as Maple, and Mathematica obtained system can be solved. Finally, $A_0, A_1, B_1, \ldots, A_j, B_j, \varrho$ are replaced in the Eq. (4).

2.2 The SEM for NLEEs

This section elucidates a systematic explanation of the SEM proposed by Kudryashov [43, 44] in 2005. First, let's consider the equation reduced to ODE (3) by wave transformation. Suppose that this equation have solutions in following form:

$$q(\xi) = \sum_{i=0}^{M} A_i \left(P(\xi)\right)^i, \tag{6}$$

where $P(\xi)$ satisfies the well-known Bernoulli and Riccati equations. Using the balancing principle, the number M can be calculated here. The coefficients $A_0, A_1, \ldots, A_M$ are the free parameters.

If the equations Bernoulli and Riccati are used, respectively, the form the solutions will have is given below.

For the Bernoulli equation:

$$P'(\xi) = \alpha P(\xi)^2 + \beta P(\xi), \tag{7}$$

where α and β are arbitrary constants. This equation is a well-known nonlinear ODE. The solution is represented as follows

$$P(\xi) = \frac{\beta(\cosh\left[\beta(\xi + C)\right] + \sinh\left[\beta(\xi + C)\right])}{1 - \alpha\,\cosh\left[\beta(\xi + C)\right] - \alpha\,\sinh\left[\beta(\xi + C)\right]}.$$

For the Riccati equation

$$P'(\xi) = \alpha P(\xi)^2 + \beta P(\xi) + d, \tag{8}$$

the solutions are represented as follows

$$P(\xi) = -\frac{\beta + \theta\,\tanh\left(\frac{1}{2}\theta(\xi + C)\right)}{2\alpha}$$

and

$$P(\xi) = -\frac{\beta + \theta\,\tanh\left(\frac{1}{2}\theta\xi\right)}{2\alpha} + \frac{\operatorname{sech}\left(\frac{\theta}{2}\xi\right)}{C\,\cosh\left(\frac{\theta}{2}\xi\right) - \frac{2\alpha}{\theta}\,\sinh\left(\frac{\theta}{2}\xi\right)}$$

where $\theta^2 = \beta^2 - 4\alpha d > 0$.

2.3 The extended G'/G-expansion method

The key steps of the proposed method [45, 46] are the following. Consider the nonlinear partial differential equation given in Eq. (3). Suppose the solution of (3) can be expressed in G'/G as follows:

$$q(\xi) = a_0 + \sum_{i=1}^{n} \left\{ a_i \left(\frac{G'}{G}\right)^i + b_i \left(\frac{G'}{G}\right)^{i-1} \sqrt{\sigma \left(1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)} \right\} \tag{9}$$

where a_0, a_i and $b_i (i = 1, \ldots, n)$ are free parameters which would be calculated subsequently, $\sigma = \pm 1$, and $G = G(\xi)$ satisfies the second-order linear ODE given below:

$$G'' + \mu G = 0, \tag{10}$$

where μ is the free parameter which would be calculated subsequently. n in Eq. (9) is positive integer can be determined by balancing principle. Collecting all terms with same powers of $\left(\frac{G'}{G}\right)^j$ and $\left(\frac{G'}{G}\right)^j \sqrt{\sigma \left(1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}$ together after imposing (9) into (3), using (10) and equating each coefficient of them to zero, a set of algebraic equations is yielded. The well-known general solution of (10) can be used for presenting the traveling wave solutions of the nonlinear partial differential Eq. (2).

3 Governing equation

The form of Van der Waals Eq. (1) [38, 39] is examined to furnish the solutions. Using $u = V(\xi)$ where $\xi = kx - lt$, we can convert (1) into a following ODE

$$\frac{(l^2 - \epsilon k^2)}{k^2} V + k^2 V'' + \eta l V' - V^3 = 0 \tag{11}$$

where "′" denotes the derivative with respect to ξ.

4 Application of ITEM

In this part, we furnish the wave solutions by using the ITEM. Applying homogeneous balancing principal V'' and V^3 in (11), we have $3j = j + 2$, $j = 1$. Hence, suppose solution of Eq. (11) as,

$$V(\xi) = A_0 + A_1 \tan\left(\frac{\phi(\xi)}{2}\right), \tag{12}$$

where $A_1 \neq 0$, A_0 are constants. Putting V, V', V'', V^3 in Eq. (11) and comparing, we get,

$$\begin{aligned} 2l^2 A_0 + k^4 A_1 \alpha\beta + k^4 A_1 \alpha c + \eta l A_1 \beta k^2 + \eta l A_1 c k^2 - 2 {A_0}^3 k^2 - 2\epsilon k^2 A_0 = 0, \\ 2\eta l A_1 \alpha k^2 + 2l^2 A_1 + k^4 A_1 c^2 - 6 {A_0}^2 A_1 k^2 - 2\epsilon k^2 A_1 - k^4 A_1 \beta^2 + 2k^4 A_1 \alpha^2 = 0, \\ 3k^4 A_1 \alpha c + \eta l A_1 c k^2 - 3k^4 A_1 \alpha\beta - 6 A_0 {A_1}^2 k^2 - \eta l A_1 \beta k^2 = 0, \\ k^4 A_1 c^2 - 2 {A_1}^3 k^2 - 2k^4 A_1 \beta c + k^4 A_1 \beta^2 = 0. \end{aligned} \tag{13}$$

Solving this system simply gives the following families of solution:

$$\left\{ \begin{array}{c} \eta = \eta, k = \sqrt{\frac{\epsilon}{-9 + 2\eta^2}}\eta \frac{1}{\sqrt{-\alpha^2 - \beta^2 + c^2}}, l = \frac{3\,\mathrm{i}\epsilon\,\eta}{\sqrt{-\alpha^2 - \beta^2 + c^2}(-9 + 2\eta^2)} \\ A_0 = \frac{\sqrt{2}}{2}\left(\frac{-\mathrm{i}\alpha^2 + \mathrm{i}c^2 - \mathrm{i}\beta^2 + \alpha\sqrt{-\alpha^2 - \beta^2 + c^2}}{\alpha^2 - c^2 + \beta^2}\right)\sqrt{\frac{\epsilon}{-9 + 2\eta^2}}\eta, A_1 = \frac{\sqrt{2}}{2}\sqrt{\frac{\epsilon}{-9 + 2\eta^2}}(-c + \beta)\eta \frac{1}{\sqrt{-\alpha^2 - \beta^2 + c^2}} \end{array} \right\} \tag{14}$$

For this coefficient set of (14), we obtained the solutions as

Family 1. When $\Delta = \alpha^2 + \beta^2 - c^2 < 0$ and $\beta - c \neq 0$, then

$$u_1(x,t) = -\frac{\sqrt{2}}{2}\sqrt{\frac{\epsilon}{-9+2\eta^2}}\eta\left(\tan\left(\left(-3\,\mathrm{i}\epsilon\,t + \sqrt{\frac{\epsilon}{-9+2\eta^2}}\left(2\eta^2-9\right)x\right)\frac{\eta}{2(-9+2\eta^2)}\right)+\mathrm{i}\right), \tag{15}$$

Family 2. When $\Delta = \alpha^2 + \beta^2 - c^2 > 0$ and $\beta - c \neq 0$, then

$$u_2(x,t) = \frac{\sqrt{2}}{2}\sqrt{\frac{\epsilon}{-9+2\eta^2}}\mathrm{i}\eta\left(-1+\tanh\left(\left(9x-2x\eta^2+3\,\mathrm{i}\epsilon\,t\right)\frac{\eta\sqrt{\epsilon}}{2\mathrm{i}\,(-9+2\eta^2)^{3/2}}\right)\right), \tag{16}$$

Family 5. When $\Delta = \alpha^2 + \beta^2 - c^2 > 0$, $\beta - c \neq 0$, and $\alpha = 0$, then

$$u_5(x,t) = -\frac{\sqrt{2}}{2}\sqrt{\frac{\epsilon}{-9+2\eta^2}}\eta\,\mathrm{i}\left(1-\tanh\left(-\mathrm{i}\,\frac{\eta\sqrt{\epsilon}}{2(-9+2\eta^2)^{3/2}}\left(9x-2x\eta^2+3\,\mathrm{i}\epsilon\,t\right)\right)\right), \tag{17}$$

Family 13. When $c = -\alpha$, then

$$u_{13}(x,t) = \frac{-\mathrm{i}\sqrt{2}\sqrt{\frac{\epsilon}{-9+2\eta^2}}\eta\,\mathrm{e}^{\mathrm{i}\eta\left(9\sqrt{\frac{\epsilon}{-9+2\eta^2}}x-2\sqrt{\frac{\epsilon}{-9+2\eta^2}}x\eta^2+3\,\mathrm{i}\epsilon\,t\right)\left(-9+2\eta^2\right)^{-1}}}{\mathrm{e}^{\mathrm{i}\eta\left(9\sqrt{\frac{\epsilon}{-9+2\eta^2}}x-2\sqrt{\frac{\epsilon}{-9+2\eta^2}}x\eta^2+3\,\mathrm{i}\epsilon\,t\right)\left(-9+2\eta^2\right)^{-1}}-\beta-\alpha}, \tag{18}$$

Family 14. When $\beta = -c$, then

$$u_{14}(x,t) = \frac{-\mathrm{i}\mathrm{e}^{\mathrm{i}\eta\left(9\sqrt{\frac{\epsilon}{-9+2\eta^2}}x-2\sqrt{\frac{\epsilon}{-9+2\eta^2}}x\eta^2+3\,\mathrm{i}\epsilon\,t\right)\left(-9+2\eta^2\right)^{-1}}c\sqrt{2}\sqrt{\frac{\epsilon}{-9+2\eta^2}}\eta}{-1+c\mathrm{e}^{\mathrm{i}\eta\left(9\sqrt{\frac{\epsilon}{-9+2\eta^2}}x-2\sqrt{\frac{\epsilon}{-9+2\eta^2}}x\eta^2+3\,\mathrm{i}\epsilon\,t\right)\left(-9+2\eta^2\right)^{-1}}}, \tag{19}$$

Family 18. When $\alpha = 0$ and $\beta = 0$,

$$u_{18}(x,t) = -\left(\mathrm{i}+\tan\left(\frac{1}{2(-9+2\eta^2)}\left(\sqrt{\frac{\epsilon}{-9+2\eta^2}}\left(-9\eta+2\eta^3\right)x-3\,\mathrm{i}\epsilon\,\eta\,t-9D+2D\eta^2\right)\right)\right) \times\frac{\sqrt{2}}{2}\sqrt{\frac{\epsilon}{-9+2\eta^2}}\eta, \tag{20}$$

5 Application of the SEM

In this section, we will exploit the SEM, which is one of the practical methods, to reach analytical solutions. We consider reduced Eq. (11). Using again homogenous principle in (11), balancing V'' and V^3, we have $3M = M + 2$, $M = 1$. Therefore, assume the solution of Eq. (11):

$$V(\xi) = a_0 + a_1 P(\xi). \tag{21}$$

Bernoulli equation:

Imposing Eq. (21) into Eq. (11) and making use of Eq. (7) and then equalizing the coefficients of P^i to zero, one gains a system of equations in terms of a_i. If the system obtained is solved using a computer program, the following sets of coefficients a_0 and a_1 are obtained:

$$\text{Set 1:}\left\{a_0 = 0, a_1 = \pm\frac{\sqrt{6}}{3}\sqrt{-\frac{\eta\,l}{\beta}}\alpha, k = \frac{\sqrt{3}}{3}\sqrt{-\frac{\eta\,l}{\beta}}, \epsilon = \frac{l\left(-9+2\eta^2\right)\beta}{3\eta}\right\}, \tag{22}$$

$$\text{Set 2:}\left\{a_0 = \pm\eta\,l\sqrt{\frac{2\beta}{3\eta\,l}}, a_1 = \pm\sqrt{\frac{2\eta\,l}{3\beta}}\alpha, k = \frac{\sqrt{3}}{3}\sqrt{\frac{\eta\,l}{\beta}}, \epsilon = -\frac{l\left(-9+2\eta^2\right)\beta}{3\eta}\right\} \tag{23}$$

As a result, solutions of Eq. (1) are

$$u_1(x,t) = \pm\sqrt{-\frac{\eta\, l}{\beta}}\alpha\beta\frac{\sqrt{6}}{3}\frac{(\cosh(\beta(\xi+C))+\sinh(\beta(\xi+C)))}{-1+\alpha\,\cosh(\beta(\xi+C))+\alpha\,\sinh(\beta(\xi+C))}, \tag{24}$$

and

$$u_2(x,t) = -\eta\, l\sqrt{\frac{2\beta}{3\eta\, l}}\left(-1+\alpha\,\cosh(\beta(\xi+C))+\alpha\,\sinh(\beta(\xi+C))\right)^{-1} \tag{25}$$

where $\xi = kx - lt$.

Riccati equation:

Substituting Eq. (21) into Eq. (11) along with Eq. (8) and then equating the coefficients of the functions P^i to zero, one gains a system of equations in term of a_i. At this stage, obtained system is solved with computer program. The following set of coefficients a_0 and a_1 is obtained:

$$\left\{\begin{array}{c} a_0 = -\dfrac{\left(-4\eta\, ld\alpha+\sqrt{-4\alpha d+\beta^2}\sqrt{\eta^2l^2}\beta+\eta\, l\beta^2\right)\sqrt{6}}{6\left(-4\alpha d+\beta^2\right)^{3/4}\sqrt[4]{\eta^2l^2}}, a_1 = -\dfrac{\sqrt{2}\sqrt[4]{\eta^2l^2}\alpha}{\sqrt{3}\sqrt[4]{-4\alpha d+\beta^2}}, \\ k = \dfrac{\sqrt{3}\sqrt[4]{\eta^2l^2}}{3\sqrt[4]{-4\alpha d+\beta^2}}, \epsilon = -\dfrac{\left(2\eta^2-9\right)\sqrt{-4\alpha d+\beta^2}\sqrt{\eta^2l^2}}{3\eta^2} \end{array}\right\}. \tag{26}$$

Hence solutions of Eq. (1) are

$$\begin{aligned} u_3(x,t) = &-\frac{\left(-4\eta\, ld\alpha+\sqrt{-4\alpha d+\beta^2}\sqrt{\eta^2l^2}\beta+\eta\, l\beta^2\right)\sqrt{6}}{6\left(-4\alpha d+\beta^2\right)^{3/4}\sqrt[4]{\eta^2l^2}} \\ &+\frac{\sqrt{2}\sqrt{3}\sqrt[4]{\eta^2l^2}\left(\beta+\sqrt{-4\alpha d+\beta^2}\,\tanh\left(\frac{\sqrt{-4\alpha d+\beta^2}}{2}(\xi+C)\right)\right)}{6\sqrt[4]{-4\alpha d+\beta^2}} \end{aligned} \tag{27}$$

and

$$\begin{aligned} u_4(x,t) = &-\frac{\left(-4\eta\, ld\alpha+\sqrt{-4\alpha d+\beta^2}\sqrt{\eta^2l^2}\beta+\eta\, l\beta^2\right)\sqrt{6}}{6\left(-4\alpha d+\beta^2\right)^{3/4}\sqrt[4]{\eta^2l^2}} \\ &-\sqrt{\frac{2}{3}}\frac{\sqrt[4]{\eta^2l^2}\alpha}{\sqrt[4]{-4\alpha d+\beta^2}}\left(-\frac{\beta+\sqrt{-4\alpha d+\beta^2}\,\tanh\left(\frac{\sqrt{-4\alpha d+\beta^2}}{2}\xi\right)}{2\alpha}\right. \\ &\left.+\frac{\operatorname{sech}\left(\frac{\sqrt{-4\alpha d+\beta^2}}{2}\xi\right)}{C\cosh\left(\frac{\sqrt{-4\alpha d+\beta^2}}{2}\xi\right)-2\frac{\alpha\sinh\left(\frac{\sqrt{-4\alpha d+\beta^2}}{2}\xi\right)}{\sqrt{-4\alpha d+\beta^2}}}\right) \end{aligned} \tag{28}$$

where $\xi = kx - lt$.

6 Application of extended G'/G-expansion method

In this subsection, we will use the method [45, 46] introduced in Subsection 2.3 to furnish the exact solutions of underlying equation. Since this algorithm is an extended and improved version of the G'/G expansion method, it is natural to expect more different solution forms. Considering the homogeneous balance principle, Eq. (11) has the solutions in the form

$$V(\xi) = a_0 + a_1\frac{G'}{G} + b_1\sqrt{\sigma\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)} + c_1\left(\frac{G'}{G}\right)^{-1} + d_1\frac{1}{\sqrt{\sigma\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}} \tag{29}$$

where a_0, a_1, b_1, c_1, d_1 are constants which would be calculated, and $\sigma = \pm 1$. The function $G = G(\xi)$ satisfies (10). Substituting (29) along with Eq. (10) into Eq. (11), and equating the coefficients of $\left(\frac{G'}{G}\right)^j$ and $\left(\frac{G'}{G}\right)^j\sqrt{\sigma\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}$ to zero, one gains a system of algebraic equations. Solving the considered system by the help of Maple the following results will be obtained:

$$\text{Set 1: } \left\{\eta = -3\frac{\sqrt{-2\mu}k^2\sqrt{2}}{l}, \epsilon = \frac{8\mu k^4 + l^2}{k^2}, a_0 = \sqrt{-2\mu}k, a_1 = \sqrt{2}k, b_1 = 0, c_1 = 0, d_1 = 0\right\}, \tag{30}$$

$$\text{Set 2: } \left\{\eta = -6\frac{\sqrt{-2\mu}k^2\sqrt{2}}{l}, \epsilon = \frac{32\mu k^4 + l^2}{k^2}, a_0 = 2\sqrt{-2\mu}k, a_1 = \sqrt{2}k, b_1 = 0, c_1 = -\mu\sqrt{2}k, d_1 = 0\right\}, \tag{31}$$

$$\text{Set 3: } \left\{\eta = 3\frac{\sqrt{-2\mu}k^2\sqrt{2}}{l}, \epsilon = \frac{8\mu k^4 + l^2}{k^2}, a_0 = \sqrt{-2\mu}k, a_1 = 0, b_1 = 0, c_1 = \mu\sqrt{2}k, d_1 = 0\right\}, \tag{32}$$

$$\text{Set 4: } \left\{\eta = 3\frac{\sqrt{-\mu}k^2}{l}, \epsilon = \frac{2\mu k^4 + l^2}{k^2}, a_0 = \frac{\sqrt{-\mu}}{\sqrt{2}}k, a_1 = -\frac{k}{\sqrt{2}}, b_1 = \sqrt{\frac{\mu}{2\sigma}}k, c_1 = 0, d_1 = 0\right\}. \tag{33}$$

When $\mu < 0$, plugging the general solutions of Eq. (10) into Eq. (29) and considering traveling wave transformation, the hyperbolic traveling wave solutions of Eq. (1) can be obtained as:

For Set 1:

$$u(x,t) = \sqrt{-2\mu}k + \frac{\sqrt{-2\mu}k\left(C_1\sinh\left(\sqrt{-\mu}(kx-lt)\right) + C_2\cosh\left(\sqrt{-\mu}(kx-lt)\right)\right)}{C_1\cosh\left(\sqrt{-\mu}(kx-lt)\right) + C_2\sinh\left(\sqrt{-\mu}(kx-lt)\right)}. \tag{34}$$

For Set 2:

$$u(x,t) = 2\sqrt{-2\mu}k + \frac{\sqrt{-2\mu}k\left(C_1\sinh\left(\sqrt{-\mu}(kx-lt)\right) + C_2\cosh\left(\sqrt{-\mu}(kx-lt)\right)\right)}{C_1\cosh\left(\sqrt{-\mu}(kx-lt)\right) + C_2\sinh\left(\sqrt{-\mu}(kx-lt)\right)} - \frac{\sqrt{-2\mu}k\left(C_1\cosh\left(\sqrt{-\mu}(kx-lt)\right) + C_2\sinh\left(\sqrt{-\mu}(kx-lt)\right)\right)}{\left(C_1\sinh\left(\sqrt{-\mu}(kx-lt)\right) + C_2\cosh\left(\sqrt{-\mu}(kx-lt)\right)\right)}. \tag{35}$$

For Set 3:

$$u(x,t) = \sqrt{-2\mu}k + \frac{\sqrt{-2\mu}k\left(C_1\cosh\left(\sqrt{-\mu}(kx-lt)\right) + C_2\sinh\left(\sqrt{-\mu}(kx-lt)\right)\right)}{\left(C_1\sinh\left(\sqrt{-\mu}(kx-lt)\right) + C_2\cosh\left(\sqrt{-\mu}(kx-lt)\right)\right)}. \tag{36}$$

For Set 4:

$$u(x,t) = \frac{\sqrt{-2\mu}k}{2} - \frac{k\sqrt{-2\mu}\left(C_1\sinh\left(\sqrt{-\mu}(kx-lt)\right) + C_2\cosh\left(\sqrt{-\mu}(kx-lt)\right)\right)}{2\left(C_1\cosh\left(\sqrt{-\mu}(kx-lt)\right) + C_2\sinh\left(\sqrt{-\mu}(kx-lt)\right)\right)} + \frac{\sqrt{2\mu}k}{2}\sqrt{1 - \frac{\left(C_1\sinh\left(\sqrt{-\mu}(kx-lt)\right) + C_2\cosh\left(\sqrt{-\mu}(kx-lt)\right)\right)^2}{\left(C_1\cosh\left(\sqrt{-\mu}(kx-lt)\right) + C_2\sinh\left(\sqrt{-\mu}(kx-lt)\right)\right)^2}}, \tag{37}$$

where C_1 and C_2 are arbitrary constants.

When $\mu > 0$, imposing the general solutions of Eq. (10) into Eq. (29) along with wave transformation, we acquire the trigonometric solutions of Eq. (1) as:

For Set 1:

$$u(x,t) = \sqrt{-2\,\mu}k + \frac{\sqrt{2}k\sqrt{\mu}\left(-C_1 \sin\left(\sqrt{\mu}(kx-lt)\right) + C_2 \cos\left(\sqrt{\mu}(kx-lt)\right)\right)}{C_1 \cos\left(\sqrt{\mu}(kx-lt)\right) + C_2 \sin\left(\sqrt{\mu}(kx-lt)\right)}. \tag{38}$$

For Set 2:

$$u(x,t) = 2\sqrt{-2\,\mu}k + \frac{\sqrt{2}k\sqrt{\mu}\left(-C_1 \sin\left(\sqrt{\mu}(kx-lt)\right) + C_2 \cos\left(\sqrt{\mu}(kx-lt)\right)\right)}{C_1 \cos\left(\sqrt{\mu}(kx-lt)\right) + C_2 \sin\left(\sqrt{\mu}(kx-lt)\right)}$$
$$-1/2\,\frac{\sqrt{2}k\sqrt{4}\sqrt{\mu}\left(C_1 \cos\left(\sqrt{\mu}(kx-lt)\right) + C_2 \sin\left(\sqrt{\mu}(kx-lt)\right)\right)}{-C_1 \sin\left(\sqrt{\mu}(kx-lt)\right) + C_2 \cos\left(\sqrt{\mu}(kx-lt)\right)}. \tag{39}$$

For Set 3:

$$u(x,t) = \sqrt{-2\,\mu}k + \frac{\sqrt{2}k\sqrt{\mu}\left(C_1 \cos\left(\sqrt{\mu}(kx-lt)\right) + C_2 \sin\left(\sqrt{\mu}(kx-lt)\right)\right)}{-C_1 \sin\left(\sqrt{\mu}(kx-lt)\right) + C_2 \cos\left(\sqrt{\mu}(kx-lt)\right)}. \tag{40}$$

For Set 4:

$$u(x,t) = \frac{\sqrt{-2\,\mu}k}{2} - \frac{\sqrt{2}k\sqrt{\mu}\left(-C_1 \sin\left(\sqrt{\mu}(kx-lt)\right) + C_2 \cos\left(\sqrt{\mu}(kx-lt)\right)\right)}{2\left(C_1 \cos\left(1/2\,\sqrt{4}\sqrt{\mu}(kx-lt)\right) + C_2 \sin\left(\sqrt{\mu}(kx-lt)\right)\right)}$$
$$+\frac{\sqrt{2\,\mu}}{2}k\sqrt{\left(1 + \frac{\left(-C_1 \sin\left(\sqrt{\mu}(kx-lt)\right) + C_2 \cos\left(\sqrt{\mu}(kx-lt)\right)\right)^2}{\left(C_1 \cos\left(\sqrt{\mu}(kx-lt)\right) + C_2 \sin\left(\sqrt{\mu}(kx-lt)\right)\right)^2}\right)}, \tag{41}$$

where C_1 and C_2 are arbitrary constants.

When $\mu = 0$, imposing the general solutions of Eq. (10) into Eq. (29), we obtain the rational function solutions of Eq. (1) as:

For Set 1:

$$u(x,t) = \sqrt{-2\,\mu}k + \frac{\sqrt{2}kC_2}{C_1 + C_2(kx-lt)}. \tag{42}$$

For Set 2:

$$u(x,t) = 2\sqrt{-2\,\mu}k + \frac{\sqrt{2}kC_2}{C_1 + C_2(kx-lt)} - \frac{\mu\sqrt{2}k\left(C_1 + C_2(kx-lt)\right)}{C_2}. \tag{43}$$

For Set 3:

$$u(x,t) = \sqrt{-2\,\mu}k + \frac{\mu\sqrt{2}k\left(C_1 + C_2(kx-lt)\right)}{C_2}, \tag{44}$$

where C_1 and C_2 are arbitrary constants.

Next, if $\mu < 0$ and $C_1 = 0, C_2 \neq 0$ in (34)–(37), we obtain the following soliton solutions:

$$u(x,t) = \frac{k\sqrt{-2\,\mu}\left(\tanh\left(\sqrt{-\mu}(kx-lt)\right) + 1\right)}{\tanh\left(\sqrt{-\mu}(kx-lt)\right)}. \tag{45}$$

$$u(x,t) = -\frac{k\left(-2\sqrt{-2\,\mu}\,\tanh\left(\sqrt{-\mu}(kx-lt)\right)\sqrt{-\mu} + \sqrt{2}\mu + \mu\sqrt{2}\left(\tanh\left(\sqrt{-\mu}(kx-lt)\right)\right)^2\right)}{\tanh\left(\sqrt{-\mu}(kx-lt)\right)\sqrt{-\mu}}. \tag{46}$$

$$u(x,t) = \frac{k\left(\sqrt{-2\,\mu}\sqrt{-\mu} + \mu\sqrt{2}\tanh\left(\sqrt{-\mu}(kx-lt)\right)\right)}{\sqrt{-\mu}}. \tag{47}$$

$$u(x,t)=\frac{k\left(2\mu\left(1-\tanh\left(\sqrt{-\mu}(kx-lt)\right)\right)+\sqrt{2\mu}\sqrt{\frac{\left(\tanh\left(\sqrt{-\mu}(kx-lt)\right)\right)^2-1}{\left(\tanh\left(\sqrt{-\mu}(kx-lt)\right)\right)^2}}\sqrt{-2\mu}\tanh\left(\sqrt{-\mu}(kx-lt)\right)\right)}{2\sqrt{-2\mu}\tanh\left(\sqrt{-\mu}(kx-lt)\right)}. \tag{48}$$

Similarly, it is clear that more solitary wave solutions can be obtained with choices of $\mu<0$ and $C_2=0, C_1\neq 0$ in (34)–(37), so we neglect this part.

Next, if $\mu<0$ and $C_1^2>C_2^2$ in (34)–(37), we obtain the following optical soliton solutions:

$$u(x,t)=\sqrt{-2\mu}k\left(1+\tanh(\sqrt{-\mu}(kx-lt)+\xi_0)\right) \tag{49}$$

$$\begin{aligned}u(x,t)=&\sqrt{-2\mu}k\left(2+\tanh(\sqrt{-\mu}(kx-lt)+\xi_0)\right)\\&-\sqrt{-2\mu}k\coth(\sqrt{-\mu}(kx-lt)+\xi_0),\end{aligned} \tag{50}$$

$$u(x,t)=\sqrt{-2\mu}k\left(1+\coth(\sqrt{-\mu}(kx-lt)+\xi_0)\right), \tag{51}$$

$$\begin{aligned}u(x,t)=&\frac{\sqrt{-2\mu}k}{2}\left(1-\tanh(\sqrt{-\mu}(kx-lt)+\xi_0)\right)\\&+\frac{\sqrt{2\mu}k}{2}\operatorname{sec h}(\sqrt{-\mu}(kx-lt)+\xi_0),\end{aligned} \tag{52}$$

where $\xi_0=\tanh^{-1}(C_2/C_1)$.

Next, if $\mu>0$ and $C_1=0, C_2\neq 0$ in (38)–(41), we obtain the following soliton solutions:

$$u(x,t)=\sqrt{-2\mu}k+\frac{\sqrt{2\mu}k}{\tan\left(\sqrt{\mu}(kx-lt)\right)}, \tag{53}$$

$$u(x,t)=2\sqrt{-2\mu}k+\frac{\sqrt{2\mu}k}{\tan\left(\sqrt{\mu}(kx-lt)\right)}-\sqrt{2\mu}k\tan\left(\sqrt{\mu}(kx-lt)\right), \tag{54}$$

$$u(x,t)=\sqrt{-2\mu}k+\sqrt{2\mu}k\tan\left(\sqrt{\mu}(kx-lt)\right), \tag{55}$$

$$\begin{aligned}u(x,t)=&\frac{\sqrt{-2\mu}k}{2}-\frac{\sqrt{2\mu}k}{2\tan\left(\sqrt{\mu}(kx-lt)\right)}\\&+\frac{\sqrt{2\mu}k}{2}\sqrt{1+\left(\tan\left(\sqrt{\mu}(kx-lt)\right)\right)^{-2}}.\end{aligned} \tag{56}$$

Similarly, it is clear that more solitary wave solutions can be obtained with choices of $\mu>0$ and $C_2=0, C_1\neq 0$ in (38)–(41), so we neglect this part.

Next, if $\mu>0$ and $C_1^2>C_2^2$ in (38)–(41), we obtain the following singular periodic waves:

$$u(x,t)=\sqrt{-2\mu}k-\sqrt{2\mu}k\tan(\sqrt{\mu}(kx-lt)-\xi_0), \tag{57}$$

$$\begin{aligned}u(x,t)=&2\sqrt{-2\mu}k-\sqrt{2}k\sqrt{\mu}\tan(\sqrt{\mu}(kx-lt)-\xi_0)\\&-\sqrt{2}k\sqrt{\mu}\cot(\sqrt{\mu}(kx-lt)-\xi_0),\end{aligned} \tag{58}$$

$$u(x,t)=\sqrt{-2\mu}k+\sqrt{2}k\sqrt{\mu}\cot(\sqrt{\mu}(kx-lt)-\xi_0), \tag{59}$$

$$u(x,t) = \frac{\sqrt{-2\mu}k}{2} + \frac{\sqrt{2}k\sqrt{\mu}}{2}\tan(\sqrt{\mu}(kx - lt) - \xi_0) + \frac{\sqrt{2\mu}}{2}k\sec(\sqrt{\mu}(kx - lt) - \xi_0), \tag{60}$$

where $\xi_0 = \tan^{-1}(C_2/C_1)$.

7 Discussion and conclusion

In this study, we used three techniques having great influence, to search for exact solution of Van der Waals equation. First, by using the ITEM, we constructed four classes of solutions and demonstrated graphs of solutions by selecting the suitable values of involved parameters. Novel characteristics of various-type solutions are obtained in Figures 1–4. Secondly, we obtained hyperbolic type solutions of (1) with applying the SEM. Different structures of solutions which obtained by using the auxiliary equations (Bernoulli and Riccati) are demonstrated in Figures 5 and 6. Finally, we implemented the extended G'/G-expansion method and obtained rational function solutions, soliton solutions and periodic singular waves. When we compare the three techniques applied in the article, namely, ITEM, SEM, and extended G'/G, we conclude that ITEM has richer traveling wave solution sets. There are nineteen solution classes depending on the coefficient conditions of the 1st order auxiliary equation considered for this method. However, all three methods are based on the principle of balancing (homogeneous balancing between the dispersive terms and the nonlinear terms) and the solution is presented in terms of the finite series expansion of the solutions of the auxiliary equation(s).Figure 1 shows a kink traveling wave solution of Eq. (15) with special parameters. Kink waves are traveling waves which rise or descend from one asymptotic state to another [47]. We demonstrated dark-bright soliton waves in Figures 2 and 5. In Figures 4 and 5, we obtained singular kink solutions. Figure 6 represents a kink traveling wave solution of Eq. (27). Contour and density graphs are also given to better understand the behavior of solutions in Figures 5 and 6. In Figures 7 and 8, the graphs of solutions (36) and (43) are given by choosing suitable values of parameters: singular kink wave. We have checked the correctness of the solutions obtained here by substituting them back into the underlying equation with the aid of Maple. When some of the solutions obtained are compared with the studies in the literature and when parameters are given arbitrary values in works in which used different methods, it can be observed that solutions with similar form are obtained [39, 41] (for example; Eq. (51) has similar mathematical form with (25) in [39] and (14) in [41]). According to our best knowledge, the remaining solutions are new. The results obtained in this study are more

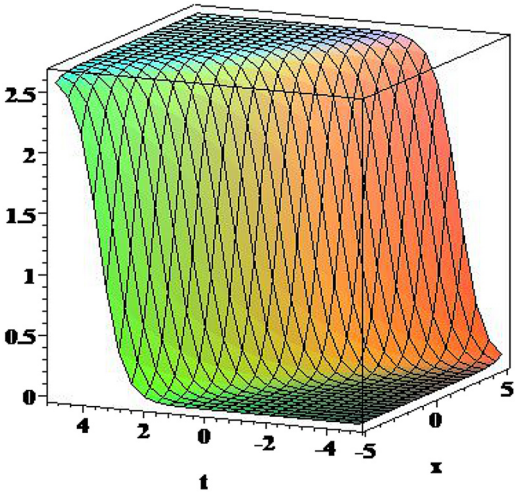

Figure 1: Plots of $u_1(x,t)$with$\epsilon = -5, \eta = 4$.

general than those obtained by many authors in the literature and more advantageous than the previously reported solutions in literature in terms of both the number of solutions and the variety of solutions.

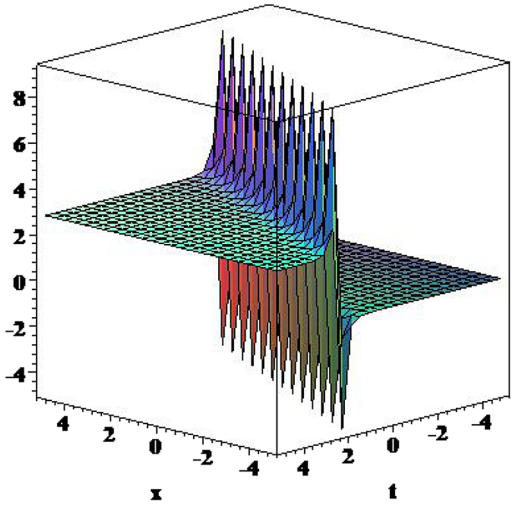

Figure 2: Plots of $u_{13}(x,t)$ with $\alpha = 0.1, \epsilon = -4, \eta = 3, \beta = 0.2$.

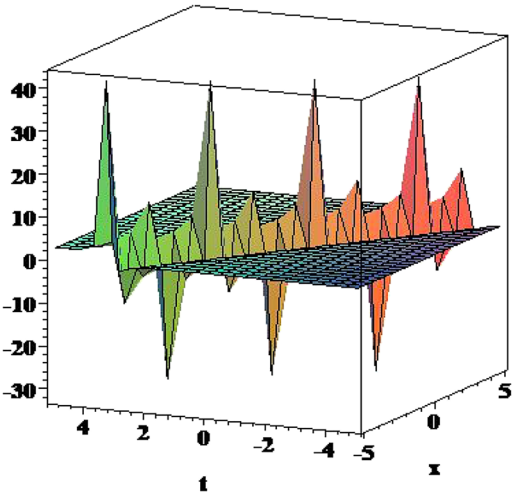

Figure 3: Plots of $u_{14}(x,t)$ with $\epsilon = -5, \eta = 4, c = 2$.

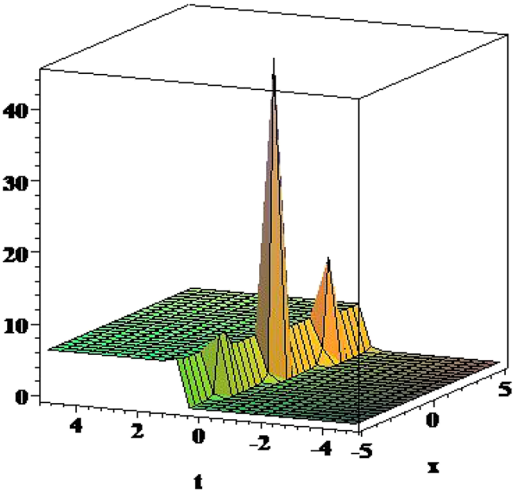

Figure 4: Plots of $u_{18}(x,t)$ with $\epsilon = 5, \eta = 2, D = 3$.

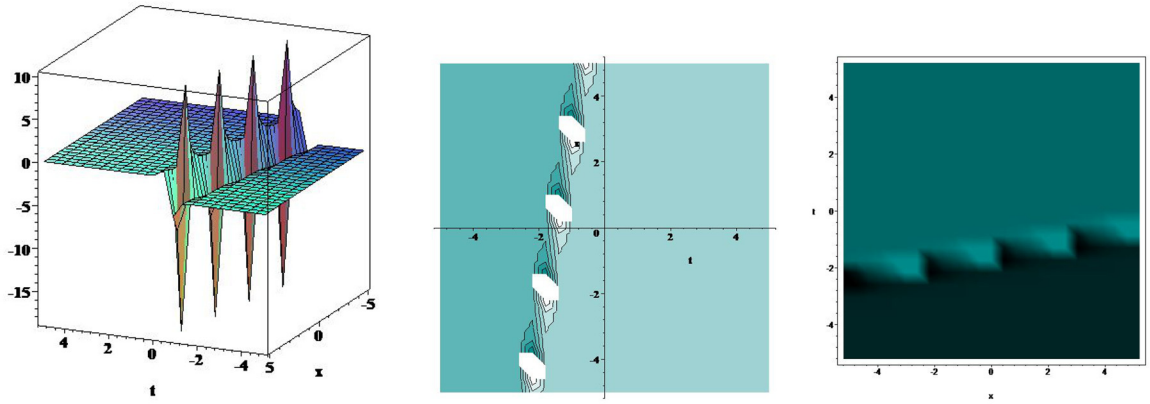

Figure 5: 3D, contour and density plots of $u_2(x,t)$ of Eq. (25) corresponding to $\alpha = 1, \beta = -1, \eta = -2, l = 6, k = 1, C = -9$, respectively.

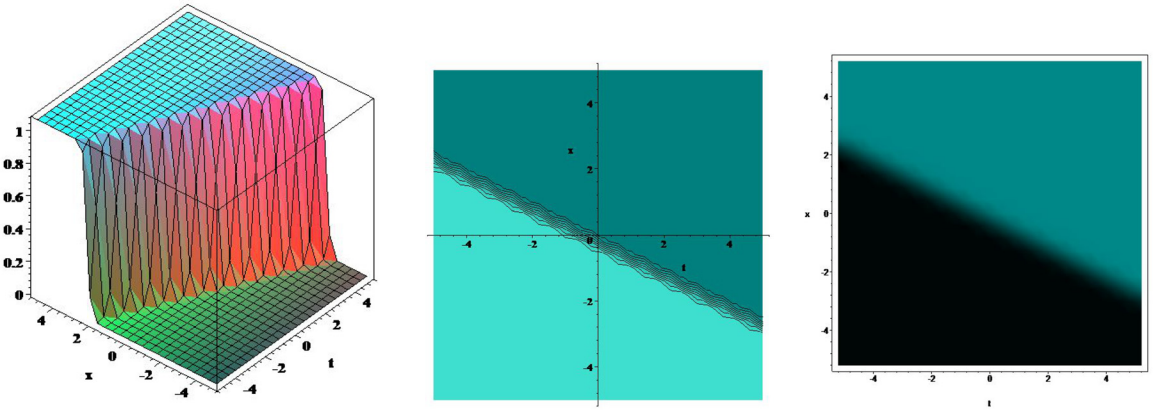

Figure 6: 3D, contour, and density plots of $u_3(x,t)$ of Eq. (27) corresponding to $\alpha = 5, d = 0.1, \beta = -2, \eta = 0.3, l = -4$, $k = 8, C = 2$, respectively.

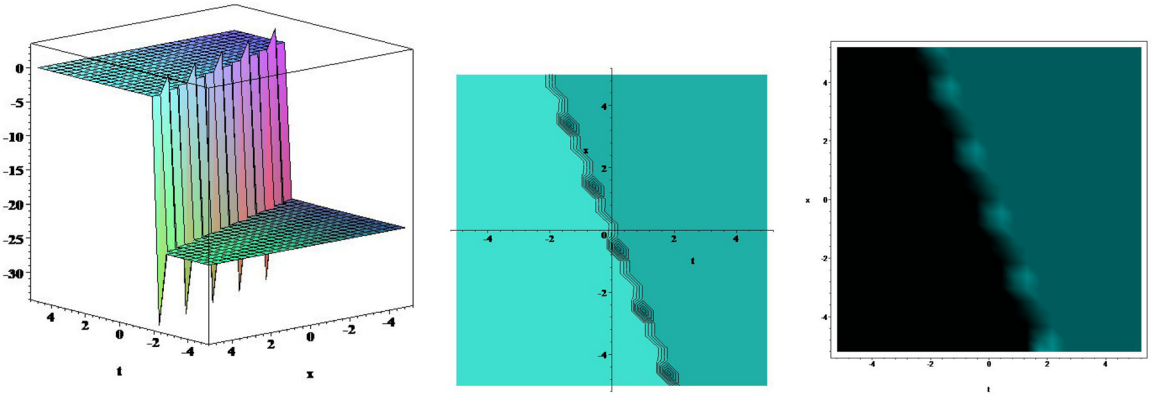

Figure 7: 3D, contour and density plots of $u(x,t)$ of Eq. (35) corresponding to $\mu = -4, k = -2, l = 5, C_1 = 2, C_2 = -1$, respectively.

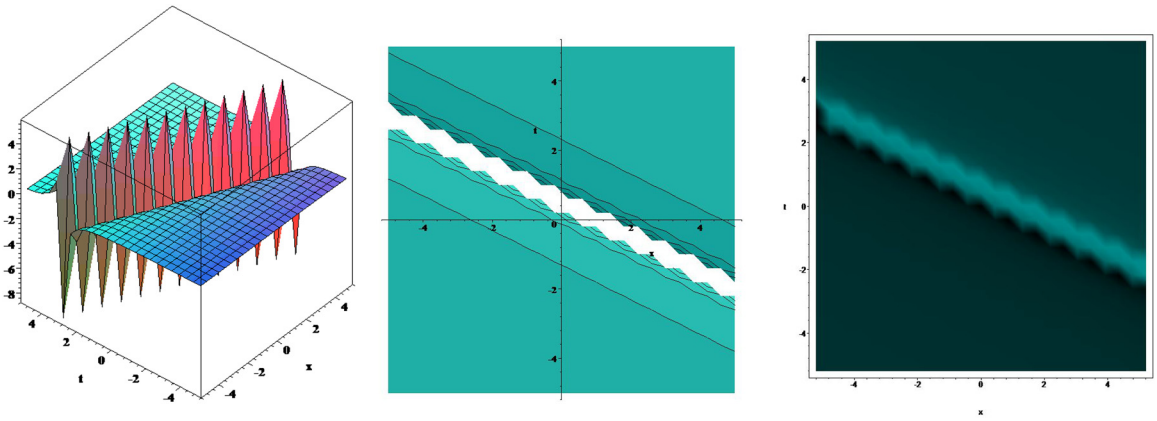

Figure 8: 3D, contour and density plots of $u(x,t)$ of Eq. (43) corresponding to $\mu = 0, k = 1, l = -2, C_1 = -1, C_2 = 1$, respectively.

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