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INCLUSION PROPERTIES FOR BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER DEFINED BY COMBINING OF FABER POLYNOMIAL EXPANSIONS AND FIBONACCI NUMBERS

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Abstract. In this present investigation, we introduce the new class $\mathcal{R}_{\Sigma,\gamma}^{\mu,\rho}\left(\widetilde{\mathfrak{p}}\right)$ of bi-univalent functions defined by using the Tremblay fractional derivative operator. Additionally, we use the Faber polynomial expansions and Fibonacci numbers to derive bounds for the general coefficient $|a_n|$ of the bi-univalent function class.

1. Introduction, Definitions and Notations

Let $\mathbb{C}$ be the complex plane, $\mathbb{U}=\{z:z\in\mathbb{C} \text{ and } |z|<1\}$ be open unit disc in $\mathbb{C}$ and let $\mathcal{A}$ denote the class of functions analytic in $\mathbb{U}$, satisfying the condition $f(0)= f'(0)-1=0$. Then each function f in $\mathcal{A}$ has the following Taylor series expansion

$$f(z)=z+a_2z^2+a_3z^3+\cdots=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1.1}$$

Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

To remind the principle of subordination between analytic functions, let the functions f and g be analytic in $\mathbb{U}$. A function f is *subordinate* to g if there exists a Schwarz function

$$\varpi(z)=\sum_{n=1}^{\infty}\mathfrak{c}_nz^n \quad (\varpi(0)=0,|\varpi(z)|<1)$$

such that

$$f(z)=g(\varpi(z)) \qquad (z\in\mathbb{U}).$$

We denote this subordination by

$$f(z)\prec g(z) \qquad (z\in\mathbb{U}).$$

For the Schwarz function $\varpi(z)$ we know that $|\mathfrak{c}_n|<1$ (see [9]).

According to the *Koebe-One Quarter Theorem*, every univalent function $f\in\mathcal{A}$ has an inverse f^{-1} satisfying $f^{-1}(f(z))=z$ $(z\in\mathbb{U})$ and $f\left(f^{-1}(w)\right)=w$ $\left(|w|<r_0(f);\ r_0(f)\geq\frac{1}{4}\right)$, where

$$\begin{aligned} g(w)=f^{-1}(w)=w\ &-a_2w^2+\left(2a_2^2-a_3\right)w^3\\ &-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\cdots. \end{aligned} \tag{1.2}$$

A function $f\in\mathcal{A}$ is called as bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief historical account and for several notable investigations of functions in the class Σ, see the pioneering work on

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this subject by Srivastava et al. [20] (see also [6, 7, 14, 15]). The interest on the estimates for the first two coefficients $|a_2|$, $|a_3|$ of the bi-univalent functions kept on by many researchers (see, for example, [4, 12, 13, 16, 21]). However, in the literature, there are only a few works (by making the use of the Faber polynomial expansions) determining the general coefficient bounds $|a_n|$ for bi-univalent functions ([5, 10, 11, 17]). The coefficient estimation problem for each of $|a_n|$ ($n \in \mathbb{N}\setminus\{1,2\}$; $\mathbb{N}=\{1,2,3,...\}$) is still an open problem.

Now, we recall to a notion of q-operators that play a major role in Geometric Function Theory. The application of the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava [18]. For the convenience, we provide some basic notation details of q-calculus which are used in this paper.

Definition 1.1. (See [19]) For a function f (analytic in a simply-connected region of $\mathbb{C}$), the fractional derivative of order ρ is stated by

$$D_z^{\rho} f(z) = \frac{1}{\Gamma(1-\rho)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\rho}} d\xi \quad (0 \leq \rho < 1)$$

and the fractional integral of order ρ is stated by

$$I_z^{\rho} f(z) = \frac{1}{\Gamma(\rho)} \int_0^z f(\xi)(z-\xi)^{\rho-1} d\xi \quad (\rho > 0).$$

Definition 1.2. (See [17]) The Tremblay fractional derivative operator of the function f is defined as

$$I_z^{\mu,\rho} f(z) = \frac{\Gamma(\rho)}{\Gamma(\mu)} z^{1-\rho} D_z^{\mu-\rho} z^{\mu-1} f(z) \quad (0 < \mu \leq 1, 0 < \rho \leq 1, \mu \geq \rho, 0 \leq \mu - \rho < 1). \quad (1.3)$$

From (1.3), we deduce that

$$I_z^{\mu,\rho} f(z) = \frac{\mu}{\rho} z + \sum_{n=2}^{\infty} \frac{\Gamma(\rho)\Gamma(n+\mu)}{\Gamma(\mu)\Gamma(n+\rho)} a_n z^n.$$

In this paper, we study the new class $\mathfrak{R}_{\Sigma,\gamma}^{\mu,\rho}\left(\widetilde{\mathfrak{p}}\right)$ of bi-univalent functions established by using the Tremblay fractional derivative operator. Further, we use the Faber polynomial expansions and Fibonacci numbers to derive bounds for the general coefficient $|a_n|$ of the bi-univalent function class.

2. Preliminaries

By utilizing the Faber polynomial expansions for functions $f \in \mathcal{A}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be stated by [2, 3]:

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...) w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!\,(n-1)!}a_2^{n-1} + \frac{(-n)!}{[2\,(-n+1)]!\,(n-3)!}a_2^{n-3}a_3 \\ &+ \frac{(-n)!}{(-2n+3)!\,(n-4)!}a_2^{n-4}a_4 \\ &+ \frac{(-n)!}{[2\,(-n+2)]!\,(n-5)!}a_2^{n-5}\left[a_5+(-n+2)\,a_3^2\right] \\ &+ \frac{(-n)!}{(-2n+5)!\,(n-6)!}a_2^{n-6}\left[a_6+(-2n+5)\,a_3a_4\right] \\ &+ \sum_{j\geq 7} a_2^{n-j}V_j, \end{aligned}$$

such that V_j $(7 \leq j \leq n)$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$. In the following, the first three terms of K_{n-1}^{-n} are stated by

$$\begin{aligned} \frac{1}{2}K_1^{-2} &= -a_2, \\ \frac{1}{3}K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4}K_3^{-4} &= -\left(5a_2^3 - 5a_2a_3 + a_4\right). \end{aligned}$$

In general, the expansion of K_n^p $(p \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\})$ is stated by

$$K_n^p = pa_n + \frac{p\,(p-1)}{2}\mathcal{G}_n^2 + \frac{p!}{(p-3)!3!}\mathcal{G}_n^3 + ... + \frac{p!}{(p-n)!n!}\mathcal{G}_n^n,$$

where $\mathcal{G}_n^p = \mathcal{G}_n^p\,(a_1, a_2, ...)$ and by [1],

$$\mathcal{G}_n^m\,(a_1, a_2, ..., a_n) = \sum_{n=1}^{\infty} \frac{m!\,(a_1)^{\delta_1} ... (a_n)^{\delta_n}}{\delta_1!...\delta_n!},$$

while $a_1 = 1$, the sum is taken over all nonnegative integers $\delta_1, ..., \delta_n$ satisfying

$$\begin{aligned} \delta_1 + \delta_2 + \ ... \ + \delta_n &= m, \\ \delta_1 + 2\delta_2 + \ ... \ + n\delta_n &= n. \end{aligned}$$

The first and the last polynomials are

$$\mathcal{G}_n^1 = a_n \qquad \mathcal{G}_n^n = a_1^n.$$

For two analytic functions $\mathfrak{u}\,(z)$, $\mathfrak{v}\,(w)$ $(\mathfrak{u}\,(0) = \mathfrak{v}\,(0) = 0,\ |\mathfrak{u}\,(z)| < 1,\ |\mathfrak{v}\,(w)| < 1)$, suppose that

$$\mathfrak{u}\,(z) = \textstyle\sum_{n=1}^{\infty} t_n z^n \quad (|z| < 1,\ z \in \mathbb{U}),$$

$$\mathfrak{v}\,(w) = \textstyle\sum_{n=1}^{\infty} s_n w^n \quad (|w| < 1,\ w \in \mathbb{U}).$$

It is well known that

$$|t_1| \leq 1, \quad |t_2| \leq 1 - |t_1|^2, \quad |s_1| \leq 1, \quad |s_2| \leq 1 - |s_1|^2. \tag{2.1}$$

Definition 2.1. A function $f \in \Sigma$ is said to be in the class

$$\mathcal{R}_{\Sigma,\gamma}^{\mu,\rho}\left(\widetilde{\mathfrak{p}}\right) \quad (\gamma \in \mathbb{C}\backslash\{0\},\ 0 < \mu \leq 1,\ 0 < \rho \leq 1,\ z, w \in \mathbb{U})$$

if the following subordination relationships are satisfied:

$$\left[1+\frac{1}{\gamma}\left(\frac{\rho\left(I_z^{\mu,\rho}f(z)\right)'}{\mu}-1\right)\right]\prec\widetilde{\mathfrak{p}}\left(z\right)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}$$

and

$$\left[1+\frac{1}{\gamma}\left(\frac{\rho\left(I_w^{\mu,\rho}g(w)\right)'}{\mu}-1\right)\right]\prec\widetilde{\mathfrak{p}}\left(w\right)=\frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2},$$

where the function g is given by (1.2) and $\tau=\frac{1-\sqrt{5}}{2}\approx-0.618$.

Remark 2.1. The function $\widetilde{\mathfrak{p}}\left(z\right)$ is not univalent in $\mathbb{U}$, but it is univalent in the disc $|z|<\frac{3-\sqrt{5}}{2}\approx 0.38$. For example, $\widetilde{\mathfrak{p}}\left(0\right)=\widetilde{\mathfrak{p}}\left(-\frac{1}{2\tau}\right)$ and $\widetilde{\mathfrak{p}}\left(e^{\pm i\arccos(1/4)}\right)=\frac{\sqrt{5}}{5}$. Also, it can be written as

$$\frac{1}{|\tau|}=\frac{|\tau|}{1-|\tau|}$$

which indicates that the number $|\tau|$ divides $[0,1]$ such that it fulfills the golden section (see for details Dziok et al. [8]).

Additionally, Dziok et al. [8] indicate a useful connection between the function $\widetilde{\mathfrak{p}}\left(z\right)$ and the Fibonacci numbers. Let $\{F_n\}$ be the sequence of Fibonacci numbers

$$F_0=0,\ F_1=1,\ F_{n+2}=F_n+F_{n+1}\ (n\in\mathbb{N}_0=\{0,1,2,\ldots\}),$$

then

$$F_n=\frac{(1-\tau)^n-\tau^n}{\sqrt{5}},\quad \tau=\frac{1-\sqrt{5}}{2}.$$

If we set

$$\begin{aligned}\widetilde{\mathfrak{p}}\left(z\right) &= 1+\sum_{n=1}^{\infty}\widetilde{\mathfrak{p}}_nz^n=1+(F_0+F_2)\tau z+(F_1+F_3)\tau^2z^2\\ &\quad+\sum_{n=3}^{\infty}(F_{n-3}+F_{n-2}+F_{n-1}+F_n)\tau^nz^n,\end{aligned}$$

then the coefficients $\widetilde{\mathfrak{p}}_n$ satisfy

$$\widetilde{\mathfrak{p}}_n=\begin{cases}\tau & (n=1)\\ 3\tau^2 & (n=2)\\ \tau\widetilde{\mathfrak{p}}_{n-1}+\tau^2\widetilde{\mathfrak{p}}_{n-2} & (n=3,4,\ldots)\end{cases}.\tag{2.2}$$

Specializing the parameters γ,μ and ρ, we state the following definitions.

Definition 2.2. For $\mu=\rho=1$, a function $f\in\Sigma$ is said to be in the class $\mathcal{R}_{\Sigma,\gamma}\left(\widetilde{\mathfrak{p}}\right)$ $(\gamma\in\mathbb{C}\backslash\{0\})$ if it satisfies the following conditions respectively:

$$\left[1+\frac{1}{\gamma}\left(f'(z)-1\right)\right]\prec\widetilde{\mathfrak{p}}\left(z\right)$$

and

$$\left[1+\frac{1}{\gamma}\left(g'(w)-1\right)\right]\prec\widetilde{\mathfrak{p}}\left(w\right),$$

where $g=f^{-1}$.

Definition 2.3. For $\gamma=\mu=\rho=1$, a function $f\in\Sigma$ is said to be in the class $\mathcal{R}_{\Sigma}\left(\widetilde{\mathfrak{p}}\right)$ if it satisfies the following conditions respectively:

$$f'(z)\prec\widetilde{\mathfrak{p}}\left(z\right)$$

and

$$g'(w)\prec\widetilde{\mathfrak{p}}\left(w\right),$$

where $g=f^{-1}$.

3. Main Result and its consequences

Theorem 3.1. *For $\gamma\in\mathbb{C}\backslash\{0\}$, let $f\in\mathcal{R}_{\Sigma,\gamma}^{\mu,\rho}\left(\widetilde{\mathfrak{p}}\right)$. If $a_m=0$ $(2\leq m\leq n-1)$, then*

$$|a_n|\leq\frac{|\gamma|\,|\tau|\,\Gamma(\mu+1)\Gamma(n+\rho)}{n\Gamma(\rho+1)\Gamma(n+\mu)}\quad(n\geq3).$$

Proof. Let f be given by (1.1). By the definition of subordination yields

$$\left[1+\frac{1}{\gamma}\left(\frac{\rho\left(I_{z}^{\mu,\rho}f(z)\right)'}{\mu}-1\right)\right]=\widetilde{\mathfrak{p}}(\mathfrak{u}(z))\tag{3.1}$$

and

$$\left[1+\frac{1}{\gamma}\left(\frac{\rho\left(I_{w}^{\mu,\rho}g(w)\right)'}{\mu}-1\right)\right]=\widetilde{\mathfrak{p}}(\mathfrak{v}(w)).\tag{3.2}$$

Now, an application of Faber polynomial expansion to the power series $\mathcal{R}_{\Sigma,\gamma}^{\mu,\rho}\left(\widetilde{\mathfrak{p}}\right)$ (e.g. see [2] or [[3], equation (1.6)]) yields

$$1+\frac{1}{\gamma}\left(\frac{\rho\left(I_{z}^{\mu,\rho}f(z)\right)'}{\mu}-1\right)=1+\frac{\Gamma(\rho+1)}{\gamma\Gamma(\mu+1)}\sum_{n=2}^{\infty}\mathcal{F}_{n-1}\left(a_{2},a_{3},...,a_{n}\right)z^{n-1},$$

where

$$\begin{aligned}\mathcal{F}_{n-1}\left(a_{2},a_{3},...,a_{n}\right)z^{n-1}&=n\tfrac{\Gamma(n+\mu)}{\Gamma(n+\rho)}\\&\times\sum_{i_1+2i_2+\cdots+(n-1)i_{(n-1)}=n-1}^{\infty}\frac{(1-(i_1+i_2+\cdots+i_{n-1}))!\left[(a_2)^{i_1}(a_3)^{i_2}...(a_n)^{i_{n-1}}\right]}{(i_1!)(i_2!)...(i_{n-1}!)}.\end{aligned}$$

In particular, the first two terms are, $\mathcal{F}_1=\frac{2(\mu+1)}{\gamma(\rho+1)}a_2,\mathcal{F}_2=\frac{3(\mu+1)(\mu+2)}{\gamma(\rho+1)(\rho+2)}a_3$.

By the same token, for its inverse map $g=f^{-1}$, it is seen that

$$\begin{aligned}1+\frac{1}{\gamma}\left(\frac{\rho\left(I_{w}^{\mu,\rho}g(w)\right)'}{\mu}-1\right)&=1+\sum_{n=2}^{\infty}\frac{\Gamma(\rho+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\rho)}\frac{n}{\gamma}\times\frac{1}{n}K_{n-1}^{-n}\left(a_{2},a_{3},...\right)w^{n-1}\\&=1+\frac{\Gamma(\rho+1)}{\gamma\Gamma(\mu+1)}\sum_{n=2}^{\infty}\mathcal{F}_{n-1}\left(b_{2},b_{3},...,b_{n}\right)w^{n-1}.\end{aligned}$$

Then, the equations (3.1) and (3.2) lead to

$$\begin{aligned}\widetilde{\mathfrak{p}}\left(\mathfrak{u}\left(z\right)\right)&=1+\widetilde{\mathfrak{p}}_{1}\mathfrak{u}(z)+\widetilde{\mathfrak{p}}_{2}(\mathfrak{u}(z))^{2}z^{2}+\cdots\\&=1+\widetilde{\mathfrak{p}}_{1}t_{1}z+\left(\widetilde{\mathfrak{p}}_{1}t_{2}+\widetilde{\mathfrak{p}}_{2}t_{1}^{2}\right)z^{2}+\cdots\\&=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\widetilde{\mathfrak{p}}_{k}\mathcal{G}_{n}^{k}\left(t_{1},t_{2},...,t_{n}\right)z^{n},\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathfrak{p}}\left(\mathfrak{v}\left(w\right)\right) &= 1+\widetilde{\mathfrak{p}}_1\mathfrak{v}(w)+\widetilde{\mathfrak{p}}_2(\mathfrak{v}(w))^2z^2+\cdots \\ &= 1+\widetilde{\mathfrak{p}}_1s_1w+\left(\widetilde{\mathfrak{p}}_1s_2+\widetilde{\mathfrak{p}}_2s_1^2\right)w^2+\cdots \\ &= 1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\widetilde{\mathfrak{p}}_k\mathcal{G}_n^k\left(s_1,s_2,...,s_n\right)w^n.\end{aligned}$$

Comparing the corresponding coefficients of (3.1) and (3.2) yields

$$\frac{\Gamma(\rho+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\rho)}\frac{n}{\gamma}a_n=\widetilde{\mathfrak{p}}_1t_{n-1},$$

$$\frac{\Gamma(\rho+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\rho)}\frac{n}{\gamma}b_n=\widetilde{\mathfrak{p}}_1s_{n-1}.$$

For $a_m=0\ \ (2\leq m\leq n-1)$, we get $b_n=-a_n$ and so

$$\frac{\Gamma(\rho+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\rho)}\frac{n}{\gamma}a_n=\widetilde{\mathfrak{p}}_1t_{n-1} \tag{3.3}$$

and

$$-\frac{\Gamma(\rho+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\rho)}\frac{n}{\gamma}a_n=\widetilde{\mathfrak{p}}_1s_{n-1}. \tag{3.4}$$

Now taking the absolute values of either of the two equations above and from (2.1), we obtain

$$|a_n|\leq\frac{|\gamma|\,|\tau|\,\Gamma(\mu+1)\Gamma(n+\rho)}{n\Gamma(\rho+1)\Gamma(n+\mu)}.$$

□

Corollary 3.2. *For $\gamma\in\mathbb{C}\backslash\{0\}$, suppose that $f\in\mathcal{R}_{\Sigma,\gamma}\left(\widetilde{\mathfrak{p}}\right)$. If $a_m=0\ \ (2\leq m\leq n-1)$, then*

$$|a_n|\leq\frac{|\gamma|\,|\tau|}{n}\quad(n\geq3).$$

Corollary 3.3. *Suppose that $f\in\mathcal{R}_{\Sigma}\left(\widetilde{\mathfrak{p}}\right)$. If $a_m=0\ \ (2\leq m\leq n-1)$, then*

$$|a_n|\leq\frac{|\tau|}{n}\quad(n\geq3).$$

Theorem 3.4. *Let $f\in\mathcal{R}_{\Sigma,\gamma}^{\mu,\rho}\left(\widetilde{\mathfrak{p}}\right)\ \ (\gamma\in\mathbb{C}\backslash\{0\})$. Then*

$$\begin{aligned}|a_2| \leq \min\Bigg\{&\frac{|\gamma|\,|\tau|}{\sqrt{\left|\frac{3\gamma(\mu+1)(\mu+2)}{(\rho+1)(\rho+2)}-\frac{12(\mu+1)^2}{(\rho+1)^2}\right||\tau|+\frac{4(\mu+1)^2}{(\rho+1)^2}}},\\ &|\tau|\sqrt{\frac{|\gamma|\,(\rho+1)(\rho+2)}{(\mu+1)(\mu+2)}}\Bigg\}\end{aligned}$$

and

$$|a_3| \leq \min\left\{\frac{|\gamma|\,\tau^2(\rho+1)(\rho+2)}{(\mu+1)(\mu+2)},\right.$$
$$\left.\frac{|\tau|}{\frac{3(\mu+1)(\mu+2)}{|\gamma|(\rho+1)(\rho+2)}}\left[1+\frac{\left[\frac{3(\mu+1)(\mu+2)|\gamma||\tau|}{(\rho+1)(\rho+2)}-\frac{4(\mu+1)^2}{(\rho+1)^2}\right]}{\left|\frac{3\gamma(\mu+1)(\mu+2)}{(\rho+1)(\rho+2)}-\frac{12(\mu+1)^2}{(\rho+1)^2}\right||\tau|+\frac{4(\mu+1)^2}{(\rho+1)^2}}\right]\right\}.$$

Proof. Replacing n by 2 and 3 in (3.3) and (3.4), respectively, we find that

$$\frac{2(\mu+1)}{\gamma(\rho+1)}a_2=\widetilde{\mathfrak{p}}_1t_1, \tag{3.5}$$

$$\frac{3(\mu+1)(\mu+2)}{\gamma(\rho+1)(\rho+2)}a_3=\widetilde{\mathfrak{p}}_1t_2+\widetilde{\mathfrak{p}}_2t_1^2, \tag{3.6}$$

$$-\frac{2(\mu+1)}{\gamma(\rho+1)}a_2=\widetilde{\mathfrak{p}}_1s_1, \tag{3.7}$$

$$\frac{3(\mu+1)(\mu+2)}{\gamma(\rho+1)(\rho+2)}(2a_2^2-a_3)=\widetilde{\mathfrak{p}}_1s_2+\widetilde{\mathfrak{p}}_2s_1^2. \tag{3.8}$$

Obviously, we obtain

$$t_1=-s_1. \tag{3.9}$$

If we add the equation (3.8) to (3.6) and use (3.9), we get

$$\frac{6(\mu+1)(\mu+2)}{\gamma(\rho+1)(\rho+2)}a_2^2=\widetilde{\mathfrak{p}}_1\left(t_2+s_2\right)+2\widetilde{\mathfrak{p}}_2t_1^2. \tag{3.10}$$

Using the value of t_1^2 from (3.5), we get

$$\left[\frac{6(\mu+1)(\mu+2)}{\gamma(\rho+1)(\rho+2)}\widetilde{\mathfrak{p}}_1^2-\frac{8(\mu+1)^2}{\gamma^2(\rho+1)^2}\widetilde{\mathfrak{p}}_2\right]a_2^2=\widetilde{\mathfrak{p}}_1^3\left(t_2+s_2\right). \tag{3.11}$$

Combining (3.11) and (2.1), we obtain

$$\begin{aligned}2\left|\frac{3(\mu+1)(\mu+2)}{\gamma(\rho+1)(\rho+2)}\widetilde{\mathfrak{p}}_1^2-\frac{4(\mu+1)^2}{\gamma^2(\rho+1)^2}\widetilde{\mathfrak{p}}_2\right||a_2|^2 &\leq \left|\widetilde{\mathfrak{p}}_1\right|^3\left(|t_2|+|s_2|\right)\\ &\leq 2\left|\widetilde{\mathfrak{p}}_1\right|^3\left(1-|t_1|^2\right)\\ &= 2\left|\widetilde{\mathfrak{p}}_1\right|^3-2\left|\widetilde{\mathfrak{p}}_1\right|^3|t_1|^2.\end{aligned}$$

It follows from (3.5) that

$$|a_2|\leq\frac{|\gamma|\,|\tau|}{\sqrt{\left|\frac{3\gamma(\mu+1)(\mu+2)}{(\rho+1)(\rho+2)}-\frac{12(\mu+1)^2}{(\rho+1)^2}\right||\tau|+\frac{4(\mu+1)^2}{(\rho+1)^2}}}. \tag{3.12}$$

Additionally, by (2.1) and (3.10)

$$\begin{aligned}\frac{6(\mu+1)(\mu+2)}{|\gamma|\,(\rho+1)(\rho+2)}|a_2|^2 &\leq \left|\widetilde{\mathfrak{p}}_1\right|\left(|t_2|+|s_2|\right)+2\left|\widetilde{\mathfrak{p}}_2\right||t_1|^2\\ &\leq 2\left|\widetilde{\mathfrak{p}}_1\right|\left(1-|t_1|^2\right)+2\left|\widetilde{\mathfrak{p}}_2\right||t_1|^2\\ &= 2\left|\widetilde{\mathfrak{p}}_1\right|+2|t_1|^2\left(\left|\widetilde{\mathfrak{p}}_2\right|-\left|\widetilde{\mathfrak{p}}_1\right|\right).\end{aligned}$$

Since $\left|\widetilde{\mathfrak{p}}_2\right| > \left|\widetilde{\mathfrak{p}}_1\right|$, we get

$$|a_2| \leq |\tau| \sqrt{\frac{|\gamma|\,(\rho+1)(\rho+2)}{(\mu+1)(\mu+2)}}.$$

Next, in order to derive the bounds on $|a_3|$, by subtracting (3.8) from (3.6), we may obtain

$$\frac{6(\mu+1)(\mu+2)}{\gamma(\rho+1)(\rho+2)}a_3 = \frac{6(\mu+1)(\mu+2)}{\gamma(\rho+1)(\rho+2)}a_2^2 + \widetilde{\mathfrak{p}}_1\,(t_2 - s_2)\,. \tag{3.13}$$

Evidently, from (3.10), we state that

$$\begin{aligned} a_3 &= \frac{\widetilde{\mathfrak{p}}_1\,(t_2+s_2) + 2\widetilde{\mathfrak{p}}_2 t_1^2}{\frac{6(\mu+1)(\mu+2)}{\gamma(\rho+1)(\rho+2)}} + \frac{\widetilde{\mathfrak{p}}_1\,(t_2-s_2)}{\frac{6(\mu+1)(\mu+2)}{\gamma(\rho+1)(\rho+2)}} \\ &= \frac{\widetilde{\mathfrak{p}}_1 t_2 + \widetilde{\mathfrak{p}}_2 t_1^2}{\frac{3(\mu+1)(\mu+2)}{\gamma(\rho+1)(\rho+2)}} \end{aligned}$$

and consequently

$$\begin{aligned} |a_3| &\leq \frac{\left|\widetilde{\mathfrak{p}}_1\right|\,|t_2| + \left|\widetilde{\mathfrak{p}}_2\right|\,|t_1|^2}{\frac{3(\mu+1)(\mu+2)}{|\gamma|(\rho+1)(\rho+2)}} \\ &\leq \frac{\left|\widetilde{\mathfrak{p}}_1\right|\left(1-|t_1|^2\right) + \left|\widetilde{\mathfrak{p}}_2\right|\,|t_1|^2}{\frac{3(\mu+1)(\mu+2)}{|\gamma|(\rho+1)(\rho+2)}} \\ &= \frac{\left|\widetilde{\mathfrak{p}}_1\right| + |t_1|^2\left(\left|\widetilde{\mathfrak{p}}_2\right| - \left|\widetilde{\mathfrak{p}}_1\right|\right)}{\frac{3(\mu+1)(\mu+2)}{|\gamma|(\rho+1)(\rho+2)}}. \end{aligned}$$

Since $\left|\widetilde{\mathfrak{p}}_2\right| > \left|\widetilde{\mathfrak{p}}_1\right|$, we must write

$$|a_3| \leq \frac{|\gamma|\,\tau^2(\rho+1)(\rho+2)}{(\mu+1)(\mu+2)}.$$

On the other hand, by (2.1) and (3.13), we have

$$\begin{aligned} \frac{6(\mu+1)(\mu+2)}{|\gamma|\,(\rho+1)(\rho+2)}\,|a_3| &\leq \frac{6(\mu+1)(\mu+2)}{|\gamma|\,(\rho+1)(\rho+2)}\,|a_2|^2 + \left|\widetilde{\mathfrak{p}}_1\right|\,(|t_2| + |s_2|) \\ &\leq \frac{6(\mu+1)(\mu+2)}{|\gamma|\,(\rho+1)(\rho+2)}\,|a_2|^2 + 2\left|\widetilde{\mathfrak{p}}_1\right|\left(1-|t_1|^2\right). \end{aligned}$$

Then, with the help of (3.5), we have

$$\frac{3(\mu+1)(\mu+2)}{|\gamma|\,(\rho+1)(\rho+2)}\,|a_3| \leq \left[\frac{3(\mu+1)(\mu+2)}{|\gamma|\,(\rho+1)(\rho+2)} - \frac{4(\mu+1)^2}{|\gamma|^2\,(\rho+1)^2\left|\widetilde{\mathfrak{p}}_1\right|}\right]|a_2|^2 + \left|\widetilde{\mathfrak{p}}_1\right|.$$

By considering (3.12), we deduce that

$$|a_3| \leq \frac{|\tau|}{\frac{3(\mu+1)(\mu+2)}{|\gamma|(\rho+1)(\rho+2)}}\left\{1 + \frac{\left[\frac{3(\mu+1)(\mu+2)|\gamma||\tau|}{(\rho+1)(\rho+2)} - \frac{4(\mu+1)^2}{(\rho+1)^2}\right]}{\left|\frac{3\gamma(\mu+1)(\mu+2)}{(\rho+1)(\rho+2)} - \frac{12(\mu+1)^2}{(\rho+1)^2}\right||\tau| + \frac{4(\mu+1)^2}{(\rho+1)^2}}\right\}.$$

□

Corollary 3.5. *Let* $f \in \mathcal{R}_{\Sigma,\gamma}\left(\widetilde{\mathfrak{p}}\right)$ $(\gamma \in \mathbb{C}\backslash\{0\})$. *Then*

$$|a_2| \leq \min\left\{\frac{|\gamma|\,|\tau|}{\sqrt{3\,|\gamma-4|\,|\tau|+4}}, |\tau|\sqrt{|\gamma|}\right\}$$

and

$$|a_3| \leq \min\left\{|\gamma|\,\tau^2, \frac{(|\gamma-4|+|\gamma|)\,|\gamma|\,\tau^2}{3\,|\gamma-4|\,|\tau|+4}\right\}.$$

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