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Distance Eccentric Connectivity Index of Graphs

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Abstract. Let $G = (V, E)$ be a connected graph. The eccentric connectivity index of G is defined by $\xi^C(G) = \sum_{u \in V(G)} deg(u)e(u)$, where $deg(u)$ and $e(u)$ denote the degree and eccentricity of the vertex u in G, respectively. In this paper, we introduce a new formulation of ξ^C that will be called the distance eccentric connectivity index of G and defined by

$$\xi^{De}(G) = \sum_{u \in V(G)} deg^{De}(u)e(u)$$

where $deg^{De}(u)$ denotes the distance eccentricity degree of the vertex u in G. The aim of this paper is to introduce and study this new topological index. The values of the eccentric connectivity index is calculated for some fundamental graph classes and also for some graph operations. Some inequalities giving upper and lower bounds for this index are obtained.

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1. Introduction

In this paper, we are concerned with only connected simple graphs $G = (V, E)$ which are finite, undirected with no loops nor multiple edges and for any two vertices u and v in G, there exists a uv-path starting from u and ending in v. Throughout this paper, we take $p = |V(G)|$ and $q = |E(G)|$. The complement of G, denoted by $\overline{G}$, is a simple graph on the same set of vertices $V(G)$ in which two vertices u and v are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The distance between any two vertices u and v in G denoted by $d(u, v)$ is the number of edges on a shortest path joining u and v. There are many interesting research areas in graph theory concentrate on the distance. Many well known graph topological indices, the most famous one being Wiener index, are defined in terms of distances between the vertices of a given graph. The eccentricity $e(u)$ of a vertex u in G is another distance based concept and defined as the maximum distance between u and any other vertex v in G, that is $e(u) = \max\{d(u, v) : v \in V(G)\}$. This notion plays an important role in the study of graphs and has many applications. The maximum and minimum eccentricity over all vertices of G are called the diameter $diam(G)$ and the radius $rad(G)$ of G, respectively. All the definitions and terminologies about graphs used in this paper are available in [13]. The path, wheel, cycle, star and complete graphs with p vertices are denoted by P_p, W_p, C_p, S_p and K_p, respectively, and $K_{r,s}$ is the complete bipartite graph on $r + s$ vertices.

In chemical graph theory, various graph invariants are used for establishing correlations of chemical structures with various physical properties, chemical reactivity, or biological activity. These graph invariants are called topological indices of (molecular) graphs. Although most of the topological graph indices are degree, distance or matrix based, there are some eccentricity-based topological indices in chemical graph theory.

In [11], the authors introduced a topological descriptor called the connectivity eccentricity index when investigating the antihypertensive activity of derivatives of N-benzylimidazole. They showed that the results obtained using the connectivity eccentricity index were better than the corresponding values obtained using Balaban's mean square distance index, [5, 6], and the accuracy of prediction was found to be about 80 percent of the active range, [11]. In [22], the authors introduced the eccentric connectivity index of a given graph, which has been employed successfully for the development of numerous mathematical models for the prediction of biological activities of diverse nature, [12, 21, 22]. The eccentric connectivity index $\xi^C(G)$ of a graph G is defined as $\xi^C(G) = \sum_{u \in V(G)} deg(u)e(u)$, [22]. When the vertex degrees are not taken into account, we obtain the total eccentricity of the graph G as $\zeta(G) = \sum_{u \in V(G)} e(u)$. We refer the reader to [3, 4, 10, 15, 17, 19, 18, 24] for explicit explanations of the related notions to eccentric connectivity index of graphs. The relation between the eccentricity indices and other topological indices has also been investigated. In [14], the relationship between eccentric connectivity index and Zagreb indices were studied and in [7], the Zagreb and multiplicative Zagreb indices of some graphs were calculated. In [20], the relation of the notion of

eccentricity and other notions related to distance in graphs are correlated with the edge fixed geodomination number of a graph.

Let $u \in V(G)$. The distance eccentricity neighborhood of u denoted by $N_{De}(u)$ is defined as

$$N_{De}(u) = \{v \in V(G) : d(u,v) = e(u)\}.$$

The cardinality of $N_{De}(u)$ is called the distance eccentricity degree of the vertex u in G and denoted by $deg^{De}(u)$, and $N_{De}[u] = N_{De}(u) \cup \{u\}$ will be called as the closed distance eccentricity neighborhood of u. Note that if u has a full degree in G, then $deg(u) = deg^{De}(u)$ and if $e(u) = 2$, then $deg_{\overline{G}}(u) = deg^{De}(u)$. We use the symbol $Q(G)$ to denote the sum of distance eccentricity degrees of all vertices in a connected graph G, that is

$$Q(G) = \sum_{u \in V(G)} deg^{De}(u).$$

The maximum and minimum distance eccentricity degrees of a vertex in G are denoted respectively by $\Delta^{De}(G)$ and $\delta^{De}(G)$:

$$\Delta^{De}(G) = \max_{u \in V} |N_{De}(u)| \quad \text{and} \quad \delta^{De}(G) = \min_{u \in V} |N_{De}(u)|,$$

see [2]. Also, we denote the set of vertices of G with eccentricity equal to α by $V_e^{\alpha}(G) \subseteq V(G)$, where $\alpha = 1, 2, \ldots, diam(G)$.

The applications of vertex eccentricity and the eccentric connectivity index of graphs motivated us to use the distance eccentricity degree instead of the normal degree which is very closed to the vertex eccentricity.

In this paper, we study this topological index by investigating some of its properties and its relations with $\xi^C(G)$ and $\zeta(G)$ of a connected graph G. One of the ways of studying graphs is graph operations. There is a number of such operations which help us to determine the properties of a large graph in terms of similar properties of the relatively smaller component graphs. For some examples, see [1, 8, 9, 16]. We compute the exact values of the distance eccentric connectivity index for some specific graphs and graph operations. Some upper and lower bounds and interesting results are obtained.

2. Some General Results on ξ^{De}

In this section, we define the distance eccentric connectivity index $\xi^{De}(G)$ of a connected graph G and compute its exact values for some specific graphs. Also, we give some bounds on ξ^{De} and its relation with some other topological indices of graph.

Definition 2.1. Let $G = (V, E)$ be a connected graph. The distance eccentric connectivity index of G is defined by

$$\xi^{De}(G) = \sum_{u \in V(G)} deg^{De}(u)e(u),$$

where $deg^{De}(u)$ denotes to the distance eccentricity degree of the vertex u in G.

The distance eccentric connectivity index $\xi^{De}(G)$ of some well-known graph classes are as follows:

Proposition 2.2.

(i) *For* $p \geq 2$, $\xi^{De}(K_p) = p(p-1)$.

(ii) *For* $p \geq 2$, $\xi^{De}(P_p) = \lceil \frac{p}{2} \rceil \Big(2p - \lceil \frac{p}{2} \rceil - 1 \Big)$.

(iii) *For* $p \geq 3$, $\xi^{De}(C_p) = \begin{cases} \frac{p^2}{2}, & p \text{ is even;} \\ 2p\lfloor \frac{p}{2} \rfloor, & p \text{ is odd.} \end{cases}$

(iv) *For* $r, s \geq 2$, $\xi^{De}(K_{r,s}) = 2\big(r(r-1) + s(s-1)\big)$.

(v) *For* $p \geq 3$, $\xi^{De}(S_p) = (p-1)(2p-3)$.

(vi) *For* $p \geq 5$, $\xi^{De}(W_p) = (p-1)(2p-7)$.

In the following, we try to determine some general bounds on ξ^{De} for a graph G and its complement $\overline{G}$.

Proposition 2.3. *Let G be a connected graph on p vertices and q edges. Then*

$$\xi^{De}(G) \leq |V_e^1(G)|(p-1) + 2\overline{q}\, diam(G),$$

where $V_e^1(G) \subseteq V(G)$ is the set of vertices of eccentricity one in G and $\overline{q}$ is the number of edges in the complement $\overline{G}$ of G. The equality holds if and only if $diam(G) \leq 2$.

Proof. Let G be a connected graph and $u \in V(G)$. If $u \in V_e^1(G)$, then $deg^{De}(u) = deg(u) = p-1$ and if $u \in V = V(G) - V_e^1(G)$, then $deg^{De}(u) \leq deg_{\overline{G}}(u)$. Thus

$$\begin{aligned} \xi^{De}(G) &= \sum_{u \in V(G)} deg^{De}(u)e(u) \\ &= |V_e^1(G)|(p-1) + \sum_{u \in V} deg^{De}(u)e(u) \\ &\leq |V_e^1(G)|(p-1) + \ diam(G) \sum_{u \in V} deg_{\overline{G}}(u) \\ &= |V_e^1(G)|(p-1) + 2\overline{q}\, diam(G). \end{aligned}$$

For the equality, it is clear that $deg^{De}(u) = deg_{\overline{G}}(u)$, for every $u \in V$ if and only if $diam(G) \leq 2$. Hence, the result holds. □

Corollary 2.4. *Let G be a (p,q)-connected graph such that $V_e^1(G) = \phi$ (G has no vertex of full degree). Then*

$$p \cdot rad(G) \leq \xi^{De}(G) \leq 2\overline{q} \cdot diam(G).$$

Theorem 2.5. *Let G be a (p,q)-connected graph with $diam(G) \geq 4$ and $\overline{G}$ be also connected. Then*

$$\xi^{De}(\overline{G}) = 4q.$$

Proof. It is well known that if a graph G is connected with $diam(G) \geq 4$ and its complement $\overline{G}$ is also connected, then $diam(\overline{G}) \leq 2$. Also, since G and $\overline{G}$ are both connected, then $e_{\overline{G}}(u) = 2$, for every $u \in V(G)$. Therefore, $deg^{De}_{\overline{G}}(u) = deg_G(u)$, for every $u \in V(G)$. Hence,

$$\xi^{De}(\overline{G}) = \sum_{u \in V(G)} deg^{De}_{\overline{G}}(u) e_{\overline{G}}(u) = 2 \sum_{u \in V(G)} deg_G(u) = 4q. \qquad \square$$

Proposition 2.6. *For any (p,q)-connected graph G, we have*

$$\xi^{De}(G) \leq (p-1)\zeta(G) - \xi^c(G) + |V_e^1(G)|(p-1).$$

Furthermore, the equality is attained if and only if $diam(G) \leq 2$.

Proof. Suppose first G has no vertices of full degree ($V_e^1(G) = \phi$). Then

$$deg^{De}_G(u) \leq deg_{\overline{G}}(u),$$

for all $u \in V(G)$. Therefore,

$$\xi^{De}(G) \leq \sum_{u \in V(G)} (p - 1 - deg_G(u)) e(u) = (p-1)\zeta(G) - \xi^c(G).$$

Suppose now $V_e^1(G) \neq \phi$. Then

$$\xi^{De}(G) - |V_e^1(G)|(p-1) \leq \sum_{u \in V(G)} (p - 1 - deg_G(u)) e(u) = (p-1)\zeta(G) - \xi^c(G),$$

which is the required result.

For the equality, it is clear that $deg^{De}(u) = deg_{\overline{G}}(u)$, for all $u \in V$ if and only if $diam(G) \leq 2$.

The first Zagreb eccentricity and the first distance eccentricity Zagreb indices of a connected graph G were defined respectively as

$$\zeta_1(G) = \sum_{u \in V(G)} \big(e(u)\big)^2$$

and

$$M_1^{De}(G) = \sum_{u \in V(G)} \big(deg^{De}(u)\big)^2,$$

[2, 23]. In the following two propositions we give relations between the distance eccentric connectivity index $\xi^{De}(G)$, the first Zagreb eccentricity index $\zeta_1(G)$ and the first distance eccentricity Zagreb index $M_1^{De}(G)$ of a connected graph G. □

Proposition 2.7. *Let G be a connected graph on $p \geq 2$ vertices. Then*

$$\xi^{De}(G) \leq p \cdot \zeta(G) - \zeta_1(G),$$

with the equality if and only if $G \cong K_p$ or $G \cong S_p$.

Proof. For $u \in V(G)$, let $d(u;i)$ be the number of vertices that have distance i from u in G, where $i = 1, 2, \cdots, e(u)$. Then we have

$$\begin{aligned} p - 1 &= deg^{De}(u) + \sum_{i=1}^{e(u)-1} d(u;i) \\ &\geq deg^{De}(u) + \sum_{i=1}^{e(u)-1} 1 = deg^{De}(u) + e(u) - 1. \end{aligned}$$

Therefore, $deg^{De}(u) \leq p - e(u)$ with the equality if and only if $e(u) = 1$ or $e(u) \geq 2$ with $d(u;1) = d(u;2) = \cdots = d(u;e(u)-1) = 1$. Then

$$\begin{aligned} \xi^{De}(G) &= \sum_{u \in V(G)} deg^{De}(u)e(u) \\ &\leq \sum_{u \in V(G)} (p - e(u))e(u) = p \cdot \zeta(G) - \zeta_1(G). \end{aligned}$$

Suppose the equality holds. Then $deg^{De}(u) = p - e(u)$ and hence either $e(u) = 1$ or $e(u) \geq 2$ with $d(u;1) = d(u;2) = \cdots = d(u;e(u)-1) = 1$ for all $u \in V(G)$.
Case 1. Suppose $e(u) = 1$ for some $u \in V(G)$. Then $deg^{De}(u) = p - 1$ and thus $e(v) = 1$ or 2 for every $v \neq u$ in G. Now, if $e(v) = 1$ for all $v \neq u$, then $G \cong K_p$ and if $e(v) = 2$ for some $v \neq u$ with $d(v;1) = 1$ and thus the vertex u is unique for all $v \neq u$ and hence $G \cong S_p$.
Case 2. Suppose now $e(u) \geq 2$ with $d(u;1) = d(u;2) = \cdots = d(u;e(u)-1) = 1$ for all $u \in V(G)$. Let $diam(G) = r$ and $P(G) = u_1u_2 \ldots u_r$ be the diametral path in G. Clearly $d(u_2;1) > 1$, a contradiction. Hence there is no connected graph G satisfying the condition $e(u) \geq 2$ with $d(u;1) = d(u;2) = \cdots = d(u;e(u)-1) = 1$ for all $u \in V(G)$. The converse is clear. □

Proposition 2.8. *Let G, $\overline{G}$ be connected graphs on p vertices and q, $\overline{q}$ edges, respectively. Then*

$$\xi^{De}(G) \leq 2p\overline{q} - M_1^{De}(G).$$

Proof. From the proof of Proposition 2.7, we have for any $u \in V(G)$, $e(u) \leq p - deg^{De}(u)$ with the equality if and only if $e(u) = 1$ or $e(u) \geq 2$ with $d(u;1) =$

$d(u;2) = \cdots = d(u;e(u)-1) = 1$. Since G and $\overline{G}$ are both connected, then $deg^{De}(u) \leq deg_{\overline{G}}(u)$ for all $u \in V(G)$. Therefore,

$$\begin{aligned}\xi^{De}(G) &\leq \sum_{u\in V(G)} deg^{De}(u)\big(p - deg^{De}(u)\big)\\ &= p\sum_{u\in V(G)} deg^{De}(u) - M_1^{De}(G)\\ &\leq 2p\overline{q} - M_1^{De}(G).\end{aligned}$$

□

3. Distance Eccentric Connectivity Index of Some Graph Operations

In this section, we compute the distance eccentric connectivity index of some graph operations.

Cartesian Product

The cartesian product of two graphs G_1 and G_2 where $|V(G_1)| = p_1$, $|V(G_2)| = p_2$ and $|E(G_1)| = q_1$, $|E(G_2)| = q_2$ is denoted by $G_1 \Box G_2$. It has the vertex set $V(G_1) \times V(G_2)$ and two vertices (u,u') and (v,v') are connected by an edge if and only if either ($[u = v$ and $u'v' \in E(G_2)]$) or ($[u' = v'$ and $uv \in E(G_1)]$). In other words, $|E(G_1 \Box G_2)| = q_1p_2 + q_2p_1$. Therefore the degree of a vertex (u,u') of $G_1 \Box G_2$ is as follows:

$$deg_{G_1\Box G_2}(u,u') = deg_{G_1}(u) + deg_{G_2}(u').$$

The cartesian product of more than two graphs is similarly denoted by

$$\prod_{i=1}^{n} G_i = G_1 \Box G_2 \Box \ldots \Box G_n = (G_1 \Box G_2 \Box \ldots \Box G_{n-1}) \Box G_n,$$

in which any two vertices $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are adjacent in $\prod_{i=1}^{n} G_i$ if and only if $u_i = v_i$, $\forall\, i \neq j$ and $u_jv_j \in E(G_j)$, where $i, j = 1, 2, \ldots, n$. If $G_1 = G_2 = \cdots = G_n = G$, we have the n-th cartesian power of G and denote it by G^n.

Lemma 3.1. *Let $G = \prod_{i=1}^{n} G_i$ and let $u = (u_1, u_2, \ldots, u_n)$ be a vertex in $V(G)$. Then*

$$e(u) = \sum_{i=1}^{n} e(u_i).$$

Lemma 3.2. *Let $G = \prod_{i=1}^{n} G_i$ and let $u = (u_1, u_2, \ldots, u_n)$ be a vertex in G. Then*

$$deg_G^{De}(u) = \prod_{i=1}^{n} deg_{G_i}^{De}(u_i).$$

As a result we have

Theorem 3.3. *Let $G = \prod_{i=1}^{n} G_i$. Then*

$$\xi^{De}(G) = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \xi^{De}(G_i) Q(G_j).$$

Proof. Let $u = (u_1, u_2, \ldots, u_n)$ be a vertex in $V(G)$. Then by Lemmas 3.1 and 3.2, we have

$$\begin{aligned}
\xi^{De}(G) &= \sum_{u\in V(G)} deg_G^{De}(u) e_G(u) \\
&= \sum_{(u_1,\ldots,u_n)\in V(G)} \prod_{i=1}^{n} deg_{G_i}^{De}(u_i) \sum_{i=1}^{n} e(u_i) \\
&= \sum_{u_1\in V(G_1)} \sum_{u_2\in V(G_2)} \cdots \sum_{u_n\in V(G_n)} \prod_{i=1}^{n} deg_{G_i}^{De}(u_i)\big(e(u_1) + \cdots + e(u_n)\big) \\
&= \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \xi^{De}(G_i) Q(G_j).
\end{aligned}$$

□

Composition

The composition $G = G_1[G_2]$ of two graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$, where $|V(G_1)| = p_1$, $|E(G_1)| = q_1$ and $|V(G_2)| = p_2$, $|E(G_2)| = q_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ where any two vertices (u, u') and (v, v') are adjacent whenever u is adjacent to v in G_1 or $u = v$ and u' is adjacent to v' in G_2. Thus

$$|E(G_1[G_2])| = q_1 p_2^2 + q_2 p_1.$$

The degree of a vertex (u, u') of $G_1[G_2]$ is as follows:

$$deg_{G_1[G_2]}(u, u') = p_2 \cdot deg_{G_1}(u) + deg_{G_2}(u').$$

Lemma 3.4. *Let $G = G_1[G_2]$ and $e(v) \neq 1$ for all $v \in V(G_1)$. Then $e_G((u, u')) = e_{G_1}(u)$.*

Lemma 3.5. *Let $G = G_1[G_2]$ and $e(v) \neq 1$ for all $v \in V(G_1)$. Then*

$$deg_G^{De}(u, u') = \begin{cases} p_2 \cdot deg_{G_1}^{De}(u) + deg_{\overline{G_2}}(u'), & \text{if } u \in V_e^2(G_1); \\ p_2 \cdot deg_{G_1}^{De}(u), & \text{otherwise.} \end{cases}$$

Theorem 3.6. *Let $G = G_1[G_2]$ and $e(v) \neq 1$ for all $v \in V(G_1)$. Then*

$$\xi^{De}(G) = p_2^2 \cdot \xi^{De}(G_1) + 4\overline{q_2} \cdot |V_e^2(G_1)|.$$

Proof. By Lemmas 3.4 and 3.5, we have

$$\begin{aligned}
\xi^{De}(G) &= \sum_{(u,u')\in V(G)} deg_G^{De}(u,u')e_G(u,u') \\
&= \sum_{u\in V(G_1)} \sum_{u'\in V(G_2)} deg_G^{De}(u,u')e_G(u,u') \\
&= \sum_{u\in V_e^2(G_1)} \sum_{u'\in V(G_2)} \big(p_2 \cdot deg_{G_1}^{De}(u) + deg_{\overline{G_2}}(u')\big)e_{G_1}(u) \\
&\quad + \sum_{u\in V(G_1)-V_e^2(G_1)} \sum_{u'\in V(G_2)} \big(p_2 \cdot deg_{G_1}^{De}(u)\big)e_{G_1}(u) \\
&= \sum_{u\in V(G_1)} \sum_{u'\in V(G_2)} \big(p_2 \cdot deg_{G_1}^{De}(u)\big)e_{G_1}(u) \\
&\quad + \sum_{u\in V_e^2(G_1)} \sum_{u'\in V(G_2)} deg_{\overline{G_2}}(u')e_{G_1}(u) \\
&= p_2^2 \cdot \xi^{De}(G_1) + 4\overline{q_2} \cdot |V_e^2(G_1)|.
\end{aligned}$$

□

Corollary 3.7. *Let $G = G_1[G_2]$ and $e(v) \neq 1$ or 2 for all $v \in V(G_1)$. Then*

$$\xi^{De}(G) = p_2^2 \cdot \xi^{De}(G_1).$$

Disjunction and Symmetric Difference

The disjunction $G_1 \vee G_2$ of two graphs G_1 and G_2 with $|V(G_1)| = p_1$, $|E(G_1)| = q_1$ and $|V(G_2)| = p_2$, $|E(G_2)| = q_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ in which (u, u') is adjacent to (v, v') whenever u is adjacent to v in G_1 or u' is adjacent to v' in G_2. So,

$$|E(G_1 \vee G_2)| = q_1 p_2^2 + q_2 p_1^2 - 2q_1 q_2.$$

The degree of a vertex (u, u') of $G_1 \vee G_2$ is as follows:

$$deg_{G_1\vee G_2}(u,u') = p_2 \cdot deg_{G_1}(u) + p_1 \cdot deg_{G_2}(u') - deg_{G_1}(u)deg_{G_2}(u').$$

Also, the symmetric difference $G_1 \oplus G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ in which (u, u') is adjacent to (v, v') whenever u is adjacent to v in G_1 or u' is adjacent to v' in G_2, but not both. From definition one can see that,

$$|E(G_1 \oplus G_2)| = q_1 p_2^2 + q_2 p_1^2 - 4q_1 q_2.$$

The degree of a vertex (u, u') of $G_1 \oplus G_2$ is as follows:

$$deg_{G_1 \oplus G_2}(u, u') = p_2 \cdot deg_{G_1}(u) + p_1 \cdot deg_{G_2}(u') - 2deg_{G_1}(u)deg_{G_2}(u').$$

The distance between any two vertices of a disjunction or a symmetric difference cannot exceed two. Thus, if $e(v) \neq 1$ for all $v \in V(G_1) \cup V(G_2)$, the eccentricity of all vertices is constant and equal to two.

Lemma 3.8. *Let G_1 and G_2 be two graphs with $e(v) \neq 1$ for all $v \in V(G_1) \cup V(G_2)$. Then*

(i) $deg^{De}_{G_1 \vee G_2}(u, u') = deg_{\overline{G_1 \vee G_2}}(u, u')$,

(ii) $deg^{De}_{G_1 \oplus G_2}(u, u') = deg_{\overline{G_1 \oplus G_2}}(u, u')$.

Proposition 3.9. *Let G_1 and G_2 be two graphs with $e(v) \neq 1$ for all $v \in V(G_1) \cup V(G_2)$. Then*

(i) $\xi^{De}(G_1 \vee G_2) = 4q_{\overline{G_1 \vee G_2}}$,

(ii) $\xi^{De}(G_1 \oplus G_2) = 4q_{\overline{G_1 \oplus G_2}}$.

Proof. The proof is straightforward. □

Join

The join $G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ such that $|V(G_1)| = p_1$, $|V(G_2)| = p_2$, $|E(G_1)| = q_1$ and $|E(G_2)| = q_2$ is the graph on the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{u_1u_2 : u_1 \in V(G_1); u_2 \in V(G_2)\}$. Hence, the join of two graphs is obtained by connecting each vertex of one graph to each vertex of the other graph, while keeping all edges of both graphs. The degree of any vertex $u \in G_1 + G_2$ is given by

$$deg_{G_1+G_2}(u) = \begin{cases} deg_{G_1}(u) + p_2, & \text{if } u \in V(G_1); \\ deg_{G_2}(u) + p_1, & \text{if } u \in V(G_2). \end{cases}$$

Actually, by using the definition of the join graph $G = \sum_{i=1}^{n} G_i$, one can see that the eccentricity of any vertex $u \in G$ does not exceed two. Therefore, if $V^1_e(G_i) = \phi$ for all $G_i \subset G$, then $e(u) = 2$ for all $u \in G$.

Lemma 3.10. *Let $G = \sum_{i=1}^{n} G_i$ and $u \in V(G)$. Then*

$$deg^{De}_G(u) = \begin{cases} |V(G)| - 1, & u \in V^1_e(G_i); \\ p_i - 1 - deg_{G_i}(u), & u \in V(G_i) - V^1_e(G_i), \text{ for } i = 1, 2, \ldots, n. \end{cases}$$

Theorem 3.11. *Let* $G = \sum_{i=1}^{n} G_i$. *Then*

$$\xi^{De}(G) = \big(|V(G)| - 1\big) \sum_{i=1}^{n} |V_e^1(G_i)| + 4 \sum_{i=1}^{n} \overline{q_i}.$$

Proof. By Lemma 3.10, we get

$$\begin{aligned}
\xi^{De}(G) &= \sum_{u \in V(G)} deg_G^{De}(u) e_G(u) \\
&= \sum_{i=1}^{n} \sum_{u \in V(G_i)} deg_G^{De}(u) e_G(u) \\
&= \sum_{i=1}^{n} \sum_{u \in V_e^1(G_i)} \big(|V(G)| - 1\big) \\
&+ 2 \sum_{i=1}^{n} \sum_{u \in V(G_i) - V_e^1(G_i)} \big(p_i - 1 - deg_{G_i}(u)\big) \\
&= \big(|V(G)| - 1\big) \sum_{i=1}^{n} |V_e^1(G_i)| + 4 \sum_{i=1}^{n} \overline{q_i}.
\end{aligned}$$

□

Corollary 3.12. *If* G_i *has no vertices of full degree* ($V_e^1(G_i) = \phi$) *for all* $i = 1, 2, \ldots, n$, *then*

$$\xi^{De}(\sum_{i=1}^{n} G_i) = 4 \sum_{i=1}^{n} \overline{q_i}.$$

Corona Product

The corona product $G_1 \circ G_2$ of two graphs G_1 and G_2 where $|V(G_1)| = p_1$, $|V(G_2)| = p_2$ and $|E(G_1)| = q_1$, $|E(G_2)| = q_2$ is the graph obtained by taking $|V(G_1)|$ copies of G_2 and joining each vertex of the i-th copy with vertex $u \in V(G_1)$. Obviously,

$$|V(G_1 \circ G_2)| = p_1(p_2 + 1)$$

and

$$|E(G_1 \circ G_2)| = q_1 + p_1(q_2 + p_2).$$

It follows from the definition of the corona product $G_1 \circ G_2$ that the degree of each vertex $u \in G_1 \circ G_2$ is given by

$$deg_{G_1 \circ G_2}(u) = \begin{cases} deg_{G_1}(u) + p_2, & \text{if } u \in V(G_1); \\ deg_{G_2}(u) + 1, & \text{if } u \in V(G_2). \end{cases}$$

Lemma 3.13. *Let $G = G_1 \circ G_2$ be a connected graph such that $G_1 \neq K_1$ and let $u \in V(G)$. Then*

$$deg_G^{De}(u) = \begin{cases} p_2 deg_{G_1}^{De}(u), & u \in V(G_1); \\ p_2 deg_{G_1}^{De}(v), & u \in V(G) - V(G_1), \end{cases}$$

where $v \in V(G_1)$ is adjacent to u.

Lemma 3.14. *Let $G = G_1 \circ G_2$ be a connected graph such that $G_1 \neq K_1$ and let $u \in V(G)$. Then*

$$e_G(u) = \begin{cases} e_{G_1}(u) + 1, & u \in V(G_1); \\ e_{G_1}(v) + 2, & u \in V(G) - V(G_1), \end{cases}$$

where $v \in V(G_1)$ is adjacent to u.

Theorem 3.15. *Let $G = G_1 \circ G_2$ be a connected graph such that $G_1 \neq K_1$. Then*

$$\xi^{De}(G)(G) = p_2\big[(p_2+1)\xi^{De}(G_1) + (2p_2+1)\,Q(G_1)\big].$$

Proof. By Lemmas 3.13 and 3.14, we have

$$\begin{aligned} \xi^{De}(G)(G) &= \sum_{u \in V(G)} deg_G^{De}(u) e_G(u) \\ &= \sum_{u \in V(G_1)} deg_G^{De}(u) e_G(u) + \sum_{v \in V(G_1)} \sum_{u \in V(G_2)} deg_G^{De}(u) e_G(u) \\ &= \sum_{u \in V(G_1)} p_2 deg_{G_1}^{De}(u)\big(e_{G_1}(u) + 1\big) \\ &\quad + \sum_{v \in V(G_1)} \sum_{u \in V(G_2)} p_2 deg_{G_1}^{De}(v)\big(e_{G_1}(v) + 2\big) \\ &= p_2\big[(p_2+1)\xi^{De}(G_1) + (2p_2+1)\,Q(G_1)\big]. \end{aligned}$$

□

Example 3.16.

(i) For any cycle C_{p_1} and any path P_{p_2} with $p_1 \geq 3$ and $p_2 \geq 1$,

$$\xi^{De}(C_{p_1} \circ P_{p_2}) = \begin{cases} 2p_1p_2\big[(p_2+1)\lfloor\frac{p_1}{2}\rfloor + 2p_2 + 1\big], & \text{if } p_1 \text{ is odd;} \\ \frac{p_1p_2}{2}\big[p_1(p_2+1) + 4p_2 + 2)\big], & \text{if } p_1 \text{ is even.} \end{cases}$$

(ii) For any two cycles C_{p_1} and C_{p_2} with $p_1, p_2 \geq 3$,

$$\xi^{De}(C_{p_1} \circ C_{p_2}) = \begin{cases} 2p_1p_2\big[(p_2+1)\lfloor\frac{p_1}{2}\rfloor + 2p_2 + 1\big], & \text{if } p_1 \text{ is odd;} \\ \frac{p_1p_2}{2}\big[p_1(p_2+1) + 4p_2 + 2)\big], & \text{if } p_1 \text{ is even.} \end{cases}$$

4. Declarations

Competing interests

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Authors' contributions

AA and AS has proposed the problem and constructed the propositions' statements. RR has done calculations with indices. AYG and INC has obtained the inequalities and also made calculations and proofs of the results on graph operations.

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