APPLICATION OF QUANTUM APPROXIMATE OPTIMIZATION ALGORITHM ON A SINGLE MACHINE SCHEDULING PROBLEM

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Abstract

Scheduling is a prevalent research area in Combinatorial Optimization and Operations Research. Many exact and heuristic solution methods are available in the literature for scheduling problems. However, most scheduling problems are NP-hard, and as the problem size increases with system constraints and an increasing number of jobs and machines, the computation time also increases exponentially, rendering classical methods impractical. Modern research explores using Quantum Computing to solve scheduling problems for faster solutions. This paper presents a quantum approximate optimization algorithm (QAOA) approach for the single-machine problem of minimizing weighted tardiness and shows that QAOA achieves optimal results.

Keywords: Quantum computing, QAOA, Single machine scheduling problem, Weighted tardiness

1. INTRODUCTION

Scheduling problems are among the most studied in the Industrial Engineering discipline. There is a myriad of papers on a wide range of problems, spanning from the simplest, such as the single-machine problem, to the more complex job-shop and open-shop problems. Due to the structure of scheduling problems, as the problem size grows, complexity quickly increases to exponential levels, rendering classical optimization techniques computationally expensive or ineffective even for small instances. Conventional techniques are no longer sufficient to solve many practically relevant scheduling problems optimally, despite the continually increasing computing power.

Quantum Computing has been a facilitator in solving intractable problems since the 2000s, providing an exponential speedup by employing quantum-mechanical principles, including superposition and entanglement (Sood et al. 2023). Quantum Computing has been applied to many areas, including cryptography, financial modeling, risk analysis, communication, and combinatorial optimization (Bova et al. 2021). In search of a faster alternative, researchers began applying quantum computing to solve scheduling problems.

In this paper, we focus on the application of quantum computing, specifically QAOA, to a single-machine scheduling problem, namely the Single Machine Weighted Tardiness Problem (SMWTP). The rest of the paper is organized as follows. Section 2 presents a review of the literature on quantum methods for single-machine scheduling. Section 3 introduces the concept of quantum computing and the QAOA algorithm. Section 4 presents the application of the QAOA algorithm on the SMWTP, and Section 5 concludes the paper.

2. THE APPLICATION OF QUANTUM ALGORITHMS TO SINGLE MACHINE SCHEDULING PROBLEMS

The single machine scheduling problem is a system in which a single machine is used for task scheduling. Most of the variants of the SMSP are solvable in polynomial time. Total Lateness is minimized by the SPT (Shortest Processing Time) dispatching rule, and Lmax (maximum lateness) and Tmax (maximum tardiness) are minimized by the EDD (Earliest Due Date) dispatching rule (Baker and Trietsch 2018). However, more complex variants of the problem also exist, such as those minimizing total tardiness, minimizing total weighted tardiness, and minimizing the number of tardy jobs. SMSP with tardiness as the performance measure is known to be NP-hard in the ordinary sense even without release dates or weights, as shown by Du and Leung (1990).

Exact algorithms have been proposed for the NP-hard versions of the SMSP. Early research focused on branch-and-bound algorithms, which remain the primary exact solution approach. These methods rely on lower bounds derived from problem relaxations, dominance rules, and partial sequence evaluation (Potts and Van Wassenhove 1985). Dynamic programming and pseudo-polynomial approaches have also been explored; however, their applicability is limited to relatively small job sets due to exponential state-space growth (Lawler 1977). Because of the complexity of exact optimization, research has largely shifted toward heuristic and metaheuristic methods for medium and large-sized instances. Early constructive heuristics include the Apparent Tardiness Cost (ATC) and Modified Due Date (MDD) heuristics (Baker and Bertrand 1982), which provide reasonable sequences but rarely achieve near-optimal performance for large problems. Metaheuristics such as tabu search, simulated annealing, genetic algorithms, particle swarm optimization, and ant colony optimization have performed favorably on the NP-hard versions of the SMSP (Bilge et al. 2007). These approaches are now considered state-of-the-art for large-scale

instances. In recent years, attention has expanded from classical methods to quantum-inspired algorithms. Although still emerging, quantum algorithms have shown promising results for NP-hard scheduling problems. There are a few papers in quantum computing literature studying the SMSP.

Yan et al. (2013) investigate the SMSP with uncertainty demand and present a hybrid algorithm, QPSO, combining particle swarm optimization (PSO) and the quantum evolutionary algorithm (QEA) to maximize the total weighting agreement index for all customer orders in a fuzzy environment. Bożejko et al. (2022) solve a small-sized SMSP with weighted tardiness using quantum annealing. In a subsequent study (Bożejko et al. 2023), the SMSP is approached with weighted tardiness by determining a lower bound on the objective function using quantum annealing, rather than solving the entire problem, to increase the problem size that can be solved with quantum algorithms. They continue this approach in subsequent studies (Bożejko et al. 2024, 2025), where they use quantum annealing within a branch-and-bound algorithm to compute lower and upper bounds for the objective function in the SMSP with weighted tardiness.

Wang and Cheng (2022) also minimize total weighted tardiness in the SMSP using Grover's quantum search and Trugenberger's quantum optimization algorithms. Grange et al. (2025) use a combination of Quantum Minimum Finding (obtained from Grover search) and dynamic programming to solve single-machine scheduling problems minimizing total weighted completion time, total weighted completion time with precedence constraints, and total weighted tardiness. They continue the same line of research by adding the total number of late jobs, the total weighted completion time with release dates, and the makespan as performance measures (Grange et al. 2024). From this brief yet exhaustive literature review, we observe that the solution to the SMSP using quantum algorithms is in its infancy.

3. QUANTUM COMPUTING AND QAOA

Quantum computing is a computing technology that originated in Quantum Physics. Classical computer processor logic is organized in bits with binary number logic. Various numbers and characters are created using 0-1s, and operations are performed using 0-1 logic. 0-1 states are defined by whether current is passing through transistors. In quantum computing, classical bit logic is replaced by qubits (quantum bits, or qubits). Qubits are also in states 0-1, but they can also exist in a superposition of states at all possible values from 0-1. Superposition is the state in which different states of an object can occur together with certain possibilities. Another fundamental principle of quantum computing is entanglement. Entangled qubits cannot be measured independently of each other; the measurement result of one instantly determines the status of the other. This property is one of the fundamental mechanisms that allow quantum algorithms to evaluate a large number of solutions simultaneously (Ajagekar et al., 2022).

Calculations that normally require many steps and 0s or 1s in a classical device can be performed in a single operation using a qubit in a quantum computer. This can make calculations much faster. However, it should be noted that although a single qubit can exist in the superposition of two classical bits, when one qubit is measured, the measurement actually results in only one classical bit (0-1). Qubits are usually formulated as 0-1 bits multiplied by certain probabilities α and β (Qubit $\psi = \alpha \mid 0 > + \beta \mid 1 >$). A qubit can exist in infinite possible states (Superposition) maintaining $\alpha^2 + \beta^2 = 1$. The relationships between qubits are not just simple logical operators like AND/OR; and are modeled with more complex gates (Sood et al., 2023). Problem solving with quantum computing and specifically QAOA is carried out with the following steps (Hadfield et al., 2019).

3.1. Preparation of the Initial State

Qubits are typically initialized either to the ground state $|0\rangle$, or to an equal superposition state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, where the $\sqrt{2}$ normalization ensures that the total probability sums to unity.

In quantum computing, the problem must first be modeled in a way that a quantum processor can handle. The transition between the mathematical models of combinatorial optimization problems and quantum algorithms is achieved through Quadratic Unconstrained Binary Optimization (QUBO) formulation. The QUBO expression is in the form of $\min_{x\in\{0,1\}^n} x^T Q x$, In this expression, $\in \mathbb{R}^{n\times n}$ is the symmetric QUBO matrix and x is the vector of binary decision variables derived from the initial mathematical model of the problem. When constructing a QUBO model, all problem constraints are relaxed and incorporated into the objective function via penalty terms.

The circuit and solution structure of each quantum algorithm varies; however, the QUBO procedure continues with the transformation of binary variables into spin variables $s_i \in \{-1,1\}$: $x_i = \frac{1-s_i}{2}$. The spin variables are integrated into the QUBO function to form an Ising energy function (Vikstal, 2023):

$$E(s) = \sum_i h_i s_i + \sum_{i<j} J_{ij} s_i s_j + \text{const} \quad (1)$$

h_i and J_{ij} are calculated using the values in Q as follows:

$$J_{ij} = \frac{1}{4} Q_{ij}, \quad h_i = \frac{1}{2}\left(Q_{ii} - \sum_j Q_{ij}\right). \quad (2)$$

The cost Hamiltonian is modeled as follows, and encodes the candidate solutions for the objective function:

$$\hat{H}_C = \sum_i h_i \hat{Z}_i + \sum_{i<j} J_{ij} \hat{Z}_i \hat{Z}_j \quad (3)$$

Beyond starter solution, QAOA proceeds with the mixer Hamiltonian:

$$\hat{H}_M = \sum_i \hat{X}_i \quad (4)$$

where $\hat{X}_i$ is the Pauli-X operator. The initial state of $\hat{H}_M$ is equal superposition as follows:

$$|+\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{z\in\{0,1\}^n} |z\rangle = \prod_{i=1}^{n} \frac{|0\rangle+|1\rangle}{\sqrt{2}} \quad (5)$$

3.2. Implementation of the Quantum Circuit

Following the QUBO transformation, quantum circuits are constructed based on the specific Quantum Algorithm to be utilized. Most algorithms use Ising Hamiltonians. The Ising Hamiltonian form facilitates the translation of problem conditions into the language of quantum mechanics while guiding the execution of the algorithm (Farhi, et al, 2014).

The circuit structures of different quantum algorithms vary in terms of initial states, costs, gates, layer structures, search space exploration patterns, and hyperparameters. At this stage, unitary gates, which constitute the algorithm's internal logic, are applied. These gates vary depending on the algorithm, and quantum circuits are prepared accordingly.

The QAOA circuit begins with the initial state $|+\rangle^{\otimes n}$. The unitary gates obtained from the cost and mixer Hamiltonians, $\hat{U}_C(\gamma_l)\& = e^{-i\gamma_l \hat{H}_C}$, and $\hat{U}_M(\beta_l)\& = e^{-i\beta_l \hat{H}_M}$, respectively, are applied in the circuit as in Figure 1. $\gamma = (\gamma_1, \dots, \gamma_p)$ ve $\beta = (\beta_1, \dots, \beta_p)$, are the hyperparameters of QAOA. The circuit can be expressed as follows (Farhi et al. 2014, Hadfield et al. 2019):

$$|\psi_p(\gamma,\beta)\rangle = \prod_{l=1}^{p} \hat{U}_M(\beta_l)\hat{U}_C(\gamma_l)|+\rangle^{\otimes n} = \hat{U}_M(\beta_p)\hat{U}_C(\gamma_p)\cdots\hat{U}_M(\beta_1)\hat{U}_C(\gamma_1)|+\rangle^{\otimes n} \quad (6)$$

QAOA tries to find $\hat{H}_C$ value by changing (γ, β) The objective function is as follows:

$$E_p(\gamma,\beta) = \langle\psi_p(\gamma,\beta)|\hat{H}_C|\psi_p(\gamma,\beta)\rangle = \sum_{z\in\{0,1\}^n} p_z(\gamma,\beta)\cdot C(z) \quad (7)$$

where $p_z(\gamma,\beta) = |\langle z|\psi_p(\gamma,\beta)\rangle|^2$ is the probability to measure bitstring z and $C(z)$ is the cost function over bitstring z.

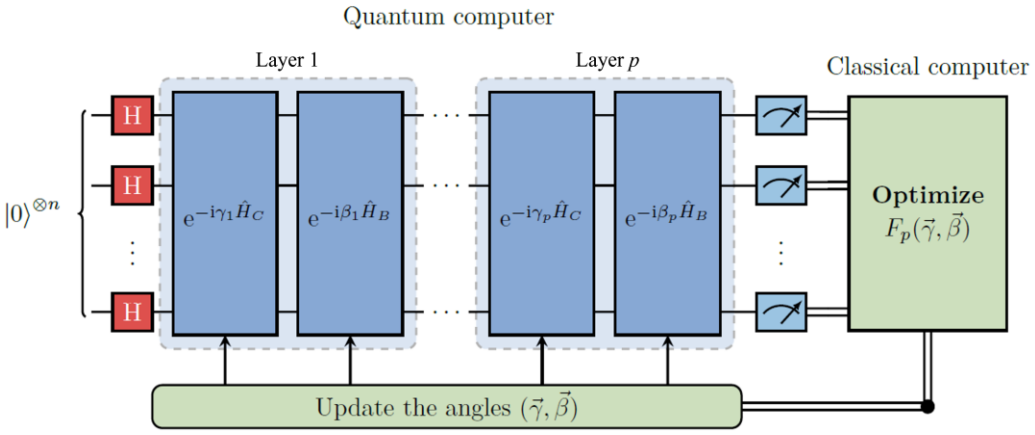


Figure 1: QAOA circuit (Vikstal 2023).

3.3. Measurement

All quantum algorithms conclude with a measurement. When measurement occurs, the superposition collapses, yielding a bit string (e.g., 11010). To determine the quantum result, many repetitions (shots) are typically performed. A circuit is usually repeated between 1000 and 10000 shots. The empirical probability is $\tilde{p}_z = \frac{N_z}{N_{\text{shots}}}$, where N_z is the number of times bitstring z is measured and N_{shots} is the number of shots. The bitstring measured most frequently is determined as the final quantum output.
Measurement results are processed on a classical computer and repeated if a loop is required. The resulting bit string is converted back into the solution to the original optimization problem—optimizing the relevant objective function —and the value is calculated classically.

4. THE APPLICATION OF QAOA TO SMWTP

The problem definition of the SMWTP is as follows: There exists a single machine m, and a set of jobs $J = \{1,2,\dots,n\}$ to be processed on machine m. Each job $j \in J = \{1,2,\dots,n\}$ is characterized by a processing time p_j, a due date d_j, and a tardiness weight w_j (Baker and Trietsch 2018). There also exists a release time for the jobs r_j. For a permutation schedule $\pi = (\pi(1), \pi(2), \dots, \pi(n))$, where $\pi(k)$ denotes the position of job j at position k, the completion time of job j at position k is denoted as $C_{\pi(k)} = \sum_{j=1}^{k} p_{\pi(j)}, k = 1,2,\dots,n$. For notational convenience, C_j denotes the completion time of job j in the schedule π. The performance measure total tardiness for a schedule can be stated as $T_w = \sum_{j=1}^{n} w_j T_j$, where $T_j = \max\{0, C_j - d_j\}$. (Baker and Trietsch 2018). The mathematical model for SMWTP is also presented in Baker and Trietsch (2018).

Applying the QAOA algorithm in Section 3, we solve a $n = 5$ job problem (Bożejko et al. 2025). The QUBO formulation has been modeled, and the circuit was designed accordingly. 8192 shots were employed. Python 3.9.13, Qiskit 0.45.0, Qiskit Aer 0.13.0, Gurobipy 11.0.0, NumPy 1.24.0, and Pandas 2.0.0. were used in problem-solving. The results in Table 1 show that QAOA can be used to solve the SMWTP and yield feasible solutions. Trials were carried out with various hyperparameter values.

Furthermore, these five problems were compared with optimal Gurobi results, as shown in Table 1, and it was observed that optimal results were obtained with QAOA. Therefore, QAOA is not only feasible but has also provided optimal results within this problem set. The qubit requirement for the solution was n^2.

Table 1: QAOA results with a small example problem set obtained from the literature

Problem	Number of qubits	QAOA Weighted Tardiness	Gurobi weighted tardiness: T_{opt}
wt5_001	25	44	44
wt5_002	25	186	186
wt5_003	25	58	58
wt5_004	25	126	126
wt5_005	25	100	100

5. CONCLUSION

Scheduling problems constitute an important class of problems in the Combinatorial Optimization literature and in the Industrial Engineering discipline. With several important performance measures and a wide variety of system assumptions, even the simplest scheduling problem can be NP-hard, making exact methods impractical, especially in real-time problems. Scheduling problems, whether single-machine or shop-type, have been attempted numerous times using classical algorithms but have rarely been studied with quantum algorithms. In this paper, we present a brief summary of the application of QAOA to the single-machine problem of minimizing total weighted tardiness. As future work, we propose applying QAOA to different variants of the scheduling problems.

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