# NONLINEAR MODELS IN OCEAN ENGINEERING: EXACT SOLUTIONS, 3D AND 2D SIMULATIONS OF THE GENERAL DRINFIEL'D-SOKOLOV-WILSON SYSTEM WITH JACOBI ELLIPTIC FUNCTIONS

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## ABSTRACT

This study highlights the crucial role of mathematics and physics in ocean engineering. In this study, the traveling wave solutions of the general Drinfiel'd-Sokolov-Wilson (DSW)-system, which is introduced as a model of water waves, are obtained and wave dynamics are investigated. Jacobi elliptic functions are valuable mathematical tools that can be applied to various aspects of ocean engineering. The methods used are effective methods to produce periodic solutions. It has been seen that the periodic solutions obtained by using Jacobi elliptic function expansions containing different Jacobi elliptic functions may be different, and some new periodic solutions can be obtained. In order to see the behaviour of the solutions obtained for the appropriate different values of the parameters, 3-dimensional simulations were made using Maple™. Their use helps engineers better understand and predict the behaviour of waves, tidal forces, and other phenomena, ultimately leading to safer and more efficient structures and systems. The stability property of the obtained solutions was tested to demonstrate the ability of the obtained solutions.

**Keywords:** DSW system, exact solutions, Jacobi elliptic function expansion method, ocean engineering

## 1. INTRODUCTION

Nonlinear evaluation (NLEEs) equations are models that emerge as a result of the study of many physical phenomena. Analytical and, in some cases, numerical solutions of these equations have an important place in various branches of science due to their concrete mathematical formulations. NLEEs can be observed in many branches of science for instance in optics, biophysics, quantum physics, quantum mechanics, chemical physics, fluid mechanics, finance, mathematical biology, even medicine, etc. The emergence of these models differently in every field has also had a positive effect on the solution methods of NLEEs, and researchers have developed many methods to obtain exact solutions of NLEEs equations. Some of these are (G'/ G) expansion method [1-2], F-expansion method [3-4], Lie symmetry approach [5-6] and so on [7-10].

The study of water waves is significant for researchers working in many branches of science. The behaviour of waves can be studied by observation or experimental means, but theoretically, mathematical modelling provides solutions to many problems in physics and engineering. Progress in this field is inevitable, with those who work in mathematics, physics, and engineering putting forth interdisciplinary studies.

Evolution differential equations have an important place in the theory of waves. Solutions of these nonlinear partial differential equation can be explained by the concept of waves, which helps us understand many physical phenomena. If the obtained solutions of the equation can be expressed with the time-dependent motion of a wave, the physical event that occurs can be explained more meaningfully.

Partial differential equation systems consisting of at least two partial differential equations are also important in modeling multiple events. Partial differential equation systems are also systems whose solutions can be obtained by the methods mentioned above.

The DSW system in dispersive water wave was introduced in 1981 by Drinfel'd and Sokolov (Drinfel'd and Sokolov, 1981), and Wilson (Wilson, 1982), which resulted in the discovery of wave phenomena that have significant applications in fluid dynamics, ocean engineering, and science. This system serves as the fundamental integrable nonlinear system that describes surface gravitational waves propagating over a horizontal seabed. The DSW equation is expressed in the following form:

$$u_t + \alpha v v_x = 0 \tag{1}$$
$$v_t + \beta u v_x + s u_x v + \eta v_{xx} = 0$$

The discovery of the DSW system and its wave phenomena has opened new avenues of research in various scientific fields and continues to pave the way for advancements in fluid dynamics and ocean engineering. Some researchers have studied this system with various methods: Bashar et al. used the new auxiliary equation (NAE) method [11], Shen et al. produced lump, soliton, and lump off solutions [12], etc.

This article is organized as follows: In Section 2, we introduced Jacobi elliptic function expansion method, In Section 3, we applied this method to the DSW system and presented various solutions. In Section 4, the stability behavior of the gained analytical solutions is studied through the properties of the Hamiltonian system. In the last chapter, the results obtained in this study are given.

## 2. MATERIAL AND METHODS

In this section, the methods used in such scientific studies are briefly introduced.

### 2.1. Jacobi Elliptic Function

Elliptic integrals were first introduced by John Wallis between 1655 and 1659. Legendre, who spent decades working on elliptic integrals, introduced normal forms of elliptic integrals that are still in use. Later, Jacobi defined elliptic functions as the inverse of elliptic integrals in 1828. Jacobi elliptic functions are obtained by inverting the first kind of elliptic integrals. For a certain constant m the function *snu* is defined with the help of the integral

$$u = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}}.$$

When this integral is reversed, than the Jacobi elliptic function *snu* is defined as $x = snu$. Similarly, *cnu* and *dnu* functions can be defined with the help of following identities:

$$sn^2\xi + cn^2\xi = 1\,, \quad dn^2\xi + m^2\, sn^2\xi = 1. \tag{2}$$

With these definitions, $sn0 = 0$, $cn0 = 1$ and $dn0 = 1$ are obtained clearly. Each Jacobi elliptic function depends on a parameter $m$ and this parameter is called the modulus of the Jacobi elliptical function. The aforementioned double periodic functions have the following properties:

$$\frac{d}{d\xi}sn(\xi) = cn(\xi)dn(\xi)\ , \ \frac{d}{d\xi}cn(\xi) = -sn(\xi)dn(\xi)\ \ \frac{d}{d\xi}dn(\xi) = -m^2cn(\xi)sn(\xi)$$

where $dn(\xi)$ is a third kind Jacobian elliptic function.

### 2.2. Jacobi Elliptic Function Expansion Method

In this section, we briefly present the Jacobi elliptic function expansion method for solving nonlinear evolution equations [13]. Consider a nonlinear evolution equation of the form:

$$P(u, u_x, u_t, u_{xx}, u_{tt}, \dots) = 0 \tag{3}$$

By using the transformation $\xi = k(x - wt)$ and $u(x,t) = U(\xi)$, the Eq.(2) can be transformed into an ordinary differential equation of the form

$$B(U, U', U'', \dots) = 0 \tag{4}$$

where $U = U(\xi), U' = \frac{dU}{d\xi}, U'' = \frac{d^2U}{d\xi^2}, \dots$ To find a periodic and solitary wave solutions of Eq.(2), we assume that $U = U(\xi)$ can be expressed as a finite series of Jacobi elliptic sine and cosine functions. Specifically, we use the following ansatz:

$$U(\xi) = \sum_{j=0}^{n} a_j sn^j(\xi) \tag{5}$$

$$U(\xi) = \sum_{j=0}^{n} b_j cn^j(\xi) \tag{6}$$

where $n, a_j, b_j \quad (j = 0,1,2,3, \dots)$ are constants. $sn(\xi) = sn(\xi, m)$ and $cn(\xi) = cn(\xi, m)$ where m (0 < m < 1) is called as modulus.

To determine the value of n, we balance the highest power nonlinear term with the highest order derivative. Thus, the highest degree of $\frac{d^pU}{d\xi^p}$ is taken as:

$$O\left(\frac{d^pU}{d\xi^p}\right) = n + p, \ p = 1,2,3, \dots$$

and the nonlinear term as

$$O\left(U^q \frac{d^pU}{d\xi^p}\right) = (q+1)n + p, \ q = 0,1,2,3, \dots$$

By substituting the ansatz (5) (or (6)) into Eq.(4) and equating the coefficients of all powers of elliptic functions to zero, we obtain a system of algebraic equations for $a_j$ (or $b_j$ ) $j = 0,1,2, \dots$. Substituting the values of $a_j$ (or $b_j$ ) back into (5) (or (6)) gives the solution of equation Eq.(4). Finally, we obtain the solution of Eq.(3) by taking $\xi = k(x - wt)$ in the solution of Eq.(4).

## 3. APPLYING THE *JACOBI ELLIPTIC FUNCTION EXPANSION METHOD* TO DSW-SYSTEM

In this section, the method mentioned in Section 2 were applied to the DSW system, which is given with Eq.(1), and the exact solutions of the system were obtained.

To apply the method given in Section 2, we first substitute the following transformation:

$$u(x,t) = u(\xi), \qquad v(x,t) = v(\xi), \qquad \xi = k(x - wt)$$

Thus, Eq. (1) can be written in the following form:

$$-kwu' + \alpha kvv' = 0 \tag{7}$$

$$-kwv' + \beta kuv' + sku'v + \eta k^3 v''' = 0$$

From the 1st equation in (7) we have

$$u = \frac{\alpha}{2w} v^2. \tag{8}$$

Using (8) in the 2nd equation in (7) and then integrating, we get undermentioned ODE:

$$-6w^2 v + \alpha(\beta + 2s)v^3 + 6\eta w k^2 v'' = 0 \tag{9}$$

Balancing the highest power nonlinear term and the highest order derivative, we obtain n=1.

*The Solutions in Terms of $sn(\xi)$:*

Since $n = 1$, using (5) the solution of (9) can be expressed as

$$v(\xi) = a_0 + a_1 sn(\xi). \tag{10}$$

Using (10) into (9) and collecting different powers of $sn(\xi)$, we obtain:

$$2\alpha s {a_0}^3 + \alpha\beta {a_0}^3 - 6w^2 a_0 = 0$$

$$12\eta k^2 w a_1 m^2 + 2\alpha s {a_1}^3 + \alpha\beta {a_1}^3 = 0$$

$$6\alpha s a_0 {a_1}^2 + 3\alpha\beta a_0 {a_1}^2 = 0$$

$$-6\eta k^2 w a_1 + 6\alpha s {a_0}^2 a_1 + 3\alpha\beta {a_0}^2 a_1 - 6\eta k^2 w a_1 m^2 - 6w^2 a_1 = 0$$

The values of the coefficients can be determined by solving the above system of equations using the Maple™ as

$$\left\{ a_0 = 0\,,\ a_1 = 2\sqrt{3}\sqrt{\frac{1+m^2}{\alpha(2s+\beta)}}\, mk^2\eta\,,\ w = -\eta k^2 - \eta k^2 m^2,\ k = k \right\}. \tag{11}$$

Substituting (11) into (10), we have

$$v_{1,m}(\xi) = 2\sqrt{3}\sqrt{\frac{1+m^2}{\alpha(2s+\beta)}}\, mk^2\eta sn(\xi, m)$$

and using (8) we have

$$u_{1,m}(\xi) = \frac{6(1+m^2)m^2k^4\eta^2}{w(2s+\beta)} sn^2(\xi, m).$$

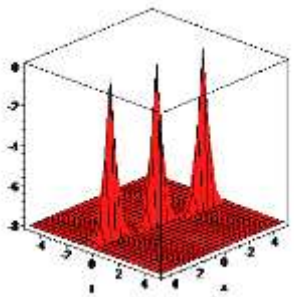
For $m \to 1$ we get $sn(\xi, m) \to tanh(\xi)$ and therefore the equations above degenerates into a solution of the DSW equation that is both periodic and exact. It can be written as

$$v_1(\xi) = 2\sqrt{2}\sqrt{3}\sqrt{\frac{1}{\alpha(2s+\beta)}}\, k^2\eta tanh(\xi)$$

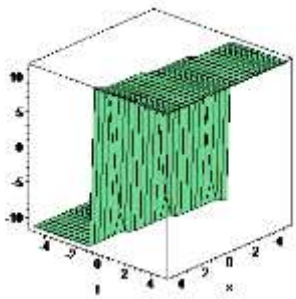
$$u_1(\xi) = -\frac{6\eta k^2 tanh^2(\xi)}{2s+\beta},$$

since $\xi = k(x - wt)$. Then the solution of the system (1)

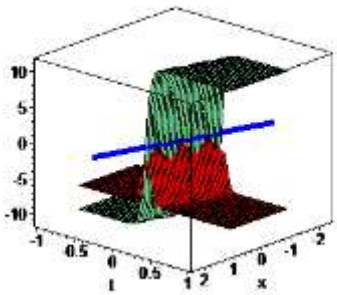
$$\left\{ v_1(x,t) = 2\sqrt{2}\sqrt{3}\sqrt{\frac{1}{\alpha(2s+\beta)}}\, k^2\eta tanh\big(k(x-wt)\big)\,,\ u_1(x,t) = -\frac{6\eta k^2 tanh^2(k(x-wt))}{2s+\beta} \right\}. \tag{12}$$



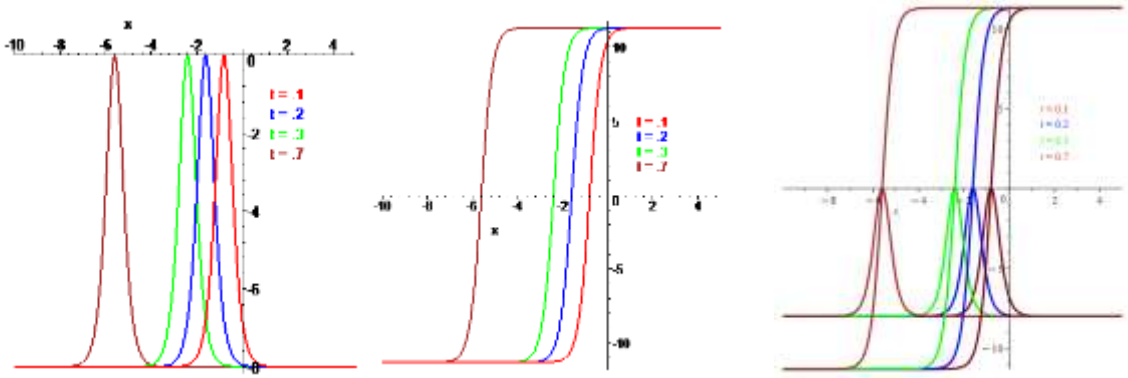
(a) $u_1(x,t)$



(b) $v_1(x,t)$



(c) $\{u_1(x,t), v_1(x,t)\}$



(d) $u_1(x,t_i)$ (e) $v_1(x,t_i)$ (f) $\{u_1(x,t_i), v_1(x,t_i)\}$

**Figure 1.** Profiles of solution (12) for $\alpha = 1, \beta = 1, s = 1, \eta = 1, k = 2, w = -8$

*The Solutions in Terms of $cn(\xi)$:*

Since $n = 1$, solution of (9) is obtained as following using (5):

$$v(\xi) = a_0 + a_1 cn(\xi) \tag{13}$$

Using (13) in (9) and collecting all the different powers of $cn(\xi)$, we obtain

$$2\alpha s {a_0}^3 + \alpha\beta {a_0}^3 - 6w^2 a_0 = 0$$

$$-12\eta k^2 w a_1 m^2 + 2\alpha s {a_1}^3 + \alpha\beta {a_1}^3 = 0$$

$$6\alpha s a_0 {a_1}^2 + 3\alpha\beta a_0 {a_1}^2 = 0$$

$$12\eta k^2 w a_1 m^2 + 6\alpha s {a_0}^2 a_1 + 3\alpha\beta {a_0}^2 a_1 - 6\eta k^2 w a_1 - 6w^2 a_1 = 0.$$

The values of the coefficients can be determined by solving the above system of equations using the Maple™ as

$$\left\{a_0 = 0,\ \ a_1 = 2\sqrt{3}\sqrt{\frac{2m^2-1}{\alpha(2s+\beta)}}mk^2\eta,\ \ w = 2\eta k^2 m^2 - \eta k^2,\ \ \ k = k\right\}. \tag{14}$$

Substituting (14) into (13), we have

$$v_{2,m}(\xi) = 2\sqrt{3}\sqrt{\frac{2m^2-1}{\alpha(2s+\beta)}}mk^2\eta cn(\xi, m)$$

and using Eq. (18) we have

$$u_{2,m}(\xi) = \frac{6(2m^2-1)m^2k^4\eta^2}{w(2s+\beta)}cn^2(\xi, m)$$

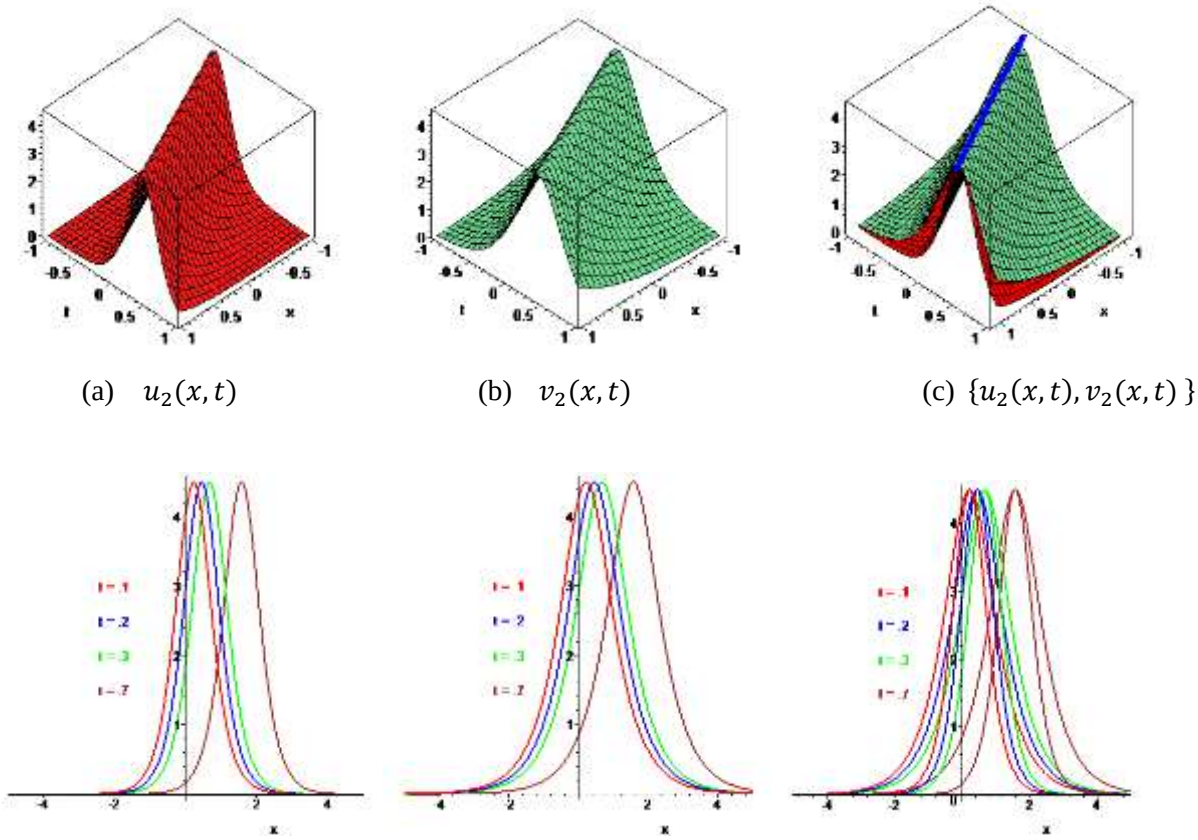
For $m \to 1$ we get $cn(\xi, m) \to sech(\xi)$ and therefore the equations above degenerates into a solution of the DSW equation that is both periodic and exact. It can be written as

$$v_2(\xi) = \frac{2\sqrt{3}k^2\eta}{\sqrt{\alpha(2s+\beta)}}sech(\xi)$$

$$u_2(\xi) = \frac{6\eta k^2}{2s+\beta}sech^2(\xi)$$

since $\xi = k(x - wt)$. Therefore the solution of the system (1) is found as

$$\left\{v_2(x,t) = \frac{2\sqrt{3}k^2\eta}{\sqrt{\alpha(2s+\beta)}}sech\big(k(x-wt)\big),\ u_2(x,t) = \frac{6\eta k^2}{2s+\beta}sech^2(k(x-wt))\right\}. \tag{15}$$



(a) $u_2(x,t)$ (b) $v_2(x,t)$ (c) $\{u_2(x,t), v_2(x,t)\}$

(d) $u_2(x,t_i)$ (e) $v_2(x,t_i)$ (f) $\{u_2(x,t_i), v_2(x,t_i)\}$

**Figure 2.** Profiles of solution (15) for $\alpha = 1, \beta = 1, s = 1, \eta = 1, k = 1.5, w = 2.25$

## 4. STABILITY PROPERTIES

Stability analysis is an important research area for nonlinear evolution equations (NLEEs). Because it helps to examine how a system responds to external influences and how it behaves over time. The stability feature obtained using the Hamilton system's properties is tested on some obtained solutions to demonstrate the model's applicability in its applications.

Here, we examine the stability property of the obtained solution with the aid of the properties of the Hamiltonian system. The momentum in the Hamiltonian system can be given by

$$W_H = \frac{1}{2}\int_{-\infty}^{\infty} v^2(\xi)d\xi \,, \qquad (16)$$

$$M_H = \frac{1}{2}\int_{-\infty}^{\infty} u^2(\xi)d\xi \,,$$

where $W_H$ and $M_H$ are the momentums, $v$ and $u$ are denotes the electric field potentials. The soliton stability conditions for the coupled model is given below,

$$\frac{\partial W}{\partial w} > 0$$

and

$$\frac{\partial M}{\partial w} > 0$$

where w is the wave velocity [14]. Inserting soliton solution Eqs. (12) into momentum equation Eqs. (16) we get the expression of the form,

$$W_H = \frac{1}{2}\int_{-5}^{5}\left(2\sqrt{2}\sqrt{3}\sqrt{\frac{1}{\alpha(2s+\beta)}}k^2\eta tanh(\xi)\right)^2 d\,\xi$$

$$M_H = \frac{1}{2}\int_{-5}^{5}\left(-\frac{6\eta k^2 tanh^2(\xi)}{2s+\beta}\right)^2 d\,\xi$$

where $\{\alpha=-1,\ \eta=1,\ \beta=-1,\ k=-1, s=1,\ w=-2\}$ and $\xi = k(x-wt)$. By solving the above integral expressions, we have

$$W_H = 1440.018794 > 0$$

$$M_H = 59.99427987 > 0$$

Hence both the equations of Eqs. (12) represents that Eq. (1) is a stable nonlinear evolution equation.

## 5. CONCLUSION

In this study, the Jacobi elliptic function expansion method was used to obtain the traveling wave solutions of the general Drinfiel'd-Sokolov-Wilson (DSW)-system introduced as a model of water waves. This method is an effective method for generating periodic solutions.

Numerical simulations of the solutions obtained from the applied method for certain parameter values were given as performed graph in Figure 1 and Figure 2. The following conclusions were reached regarding the solutions:

Unlike other studies, in this study, the graphical simulations of the solution function of the system $(u, v)$ are shown on the same axes, in different colours, and the intersection points are coloured differently (Figure 1-Figure 2). When we look at Figure 1, we see that the soliton solution and the kink solution, which are the graphical simulations of (12), form the solution of the system on the same axes. When we look at Figure 2, we see that double-kink soliton solutions, which are the graphical simulations of (15), form the solution of the system on the same axes.

Also, stability analysis was performed, which helps to examine how a system responds to external influences and how it behaves over time. The stability property of a obtained solution is discussed using momentum in the Hamiltonian system.

Each of the exact solutions obtained was written in the DSW system and the accuracy of the solution was checked with the Maple program.

We think that the new wave solutions obtained as a result of the interaction of water waves and long waves by applying Jacobi elliptical methods from the DSW system will have a significant impact in the field of ocean engineering. We believe that this study will be useful for those working in the fields of physics and engineering on the interpretation of ocean waves, the physics of underwater sounds, and how to make sense of underwater sounds.

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