On Leap Eccentric Connectivity Index of Graphs

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Abstract

For a graph $G = (V, E)$, the second degree of a vertex v in a graph G is the number of its second neighbors, that is the number of vertices in G having distance 2 to v. Zagreb indices form an important class of topological graph indices and there are numerous papers dealing with theory and applications of them. Recently, the first, second and third leap Zagreb indices have been introduced and they are respectively equal to the sum of squares of second degrees of all vertices of G, the sum of products of second degrees of the end vertices of edges in G and the sum of products of first degrees and second degrees of the vertices of G. In this paper, we introduce the leap eccentric connectivity index $L\xi^c(G)$ of a graph G which is defined as the sum of products of the second degreeswas and the eccentricity of every vertex in G. The exact values of $L\xi^c(G)$ for some well-known graph classes are obtained. Some properties and bounds on leap eccentric connectivity index are obtained.

Key words: Eccentricity, vertex degree, leap Zagreb indices, eccentricity index.
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1 Introduction

In this paper, we are concerned only with simple graphs, i.e., finite graphs having no loops, multiple or directed edges. Let $G = (V, E)$ be such a graph with vertex set $V(G)$ and edge set $E(G)$. As usual, we denote the number of vertices and edges in a graph G by n and m, respectively. The distance $d_G(u, v)$ between any two vertices u and v of a graph G is equal to the length of (number of edges in) a shortest path connecting them. For a vertex $v \in V(G)$ and a positive integer k, the open k-neighborhood of v in G is denoted by $N_k(v)$ and is defined as

$N_k(v) = \{u \in V(G) : d_G(u, v) = k\}$. The k-distance degree of a vertex v in G is denoted by $d_k(v)$ and is defined as the number of k-neighbours of the vertex v in G, i.e., $d_k(v) = |N_k(v)|$. It is clear that $d_1(v) = d(v)$ for every $v \in V(G)$. For a vertex v of G, the eccentricity of v is $e(v) = \max\{d_G(v, u) : u \in V(G)\}$. The diameter of G is $diam(G) = \max\{e(v) : v \in V(G)\}$ and the radius of G is $rad(G) = \min\{e(v) : v \in V(G)\}$. The join $G + H$ of two graphs G and H is a graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), \ \ v \in V(H)\}$. For any terminology or notation not mentioned here, we refer to [18].

A topological index of a graph is an invariant number calculated from a graph usually representing a molecule and has applications in chemistry. The Zagreb indices have been introduced by Gutman and Trinajstic, [14], in 1972 and elaborated in [16]. They are defined by

$$M_1(G) = \sum_{v \in V(G)} d_1^2(v) \ \ and \ \ M_2 = \sum_{uv \in E(G)} d_1(u)d_1(v).$$

For properties of these two Zagreb indices, see [6], [9], [16], [17], [24], [39] and the papers therein.

In recent years, some novel variants of these two ordinary Zagreb indices are introduced and studied, such as Zagreb coincides, [2], [15], multiplicative Zagreb indices, [14], [31], [39], multiplicative sum Zagreb index, [12], and multiplicative Zagreb coincides, [38], and etc. In [1, 3, 4, 5, 10, 11, 8, 25, 26, 32, 33, 35, 34, 42, 41, 43, 44], several properties of Zagreb indices have been intensively studied.

The total eccentricity of a graph, denoted by $\theta(G)$, is the sum of all the eccentricities of G, [7], i.e.

$$\theta(G) = \sum_{v \in V(G)} e(v).$$

Recently several graph invariants are defined based on vertex eccentricities in a large number of studies. Analogously to Zagreb indices, Ghorbani et. al., [13], and Vukičević et. al., [36], defined the Zagreb eccentricity indices by replacing degrees by the eccentricity of the vertices. Thus the first and second Zagreb eccentricity indices of a graph G are defined by

$$E_1(G) = \sum_{v \in V(G)} e(v)^2,$$

and

$$E_2(G) = \sum_{uv \in E(G)} e(u)e(v).$$

The lower and upper bounds of n-vertex trees with fixed diameter and matching number and also the extremal trees with respect to Zagreb eccentricity indices were studied by Xing et. al., [37], and recently, Das et. al., in [10], presented some properties on upper and lower bounds of Zagreb eccentricity indices and also characterize the extremal graphs.

In 2017, Naji et. al., [23], had introduced three new distance-degree-based topological indices depending on the second degrees of vertices (number of their second neighbours), and are so-called leap Zagreb indices of graph G and are respectively defined by

$$LM_1(G) = \sum_{v \in V(G)} d_2^2(v),$$

$$LM_2(G) = \sum_{uv \in E(G)} d_2(u) d_2(v),$$

$$LM_3(G) = \sum_{v \in V(G)} d(v) d_2(v).$$

In [22], the first leap Zagreb index of some graphs operations are computed. In [3], the expressions for these three leap Zagreb indices of generalized xyz point line transformation graphs $T^{xyz}(G)$ are obtained when $z = 1$. For properties and more details of leap Zagreb indices, we refer to [3], [23], [22], [29], [28] and [30].

Another useful eccentricity and degree based topological index is the eccentric connectivity index which was first introduced by Sharma et al., [27], and is defined as

$$\xi^c(G) = \sum_{v \in V(G)} d(v) e(v) = \sum_{uv \in E(G)} [\varepsilon(u) + e(v)].$$

The index $\xi^c(G)$ has been successfully used for mathematical models of biological activities of diverse nature. However, the investigation of its mathematical properties started only recently. For further information on eccentric connectivity index, see [19], [20], [45].

The following fundamental results which will be used in many of our subsequent considerations can be found in [40]:

Lemma 1.1. *[40] Let G be a connected graph with n vertices and m edges. Then*

$$d_2(v) \leq \left(\sum_{u \in N_1(v)} d_1(u) \right) - d_1(v).$$

Equality holds if and only if G is a $\{C_3, C_4\}$-free graph.

By Lemma 1.1, if G is a (C_3, C_4)-free graph, we obtain

$$\sum_{v \in V(G)} d_2(v) = M_1 - 2m.$$

Lemma 1.2. *[40] Let G be a connected graph with n vertices. Then for any vertex $v \in V(G)$, we have*

$$d_2(v) \leq n + 1 - d_1(v) - e(v).$$

Observation 1.3. *Let G be a connected graph with n vertices. Then for any vertex $v \in V(G)$, we have*

$$d_2(v) \leq n - 1 - d_1(v).$$

Equality holds if and only if G has diameter at most two.

Corollary 1.4. *Let G be a $\{C_3, C_4\}$-free k-regular graph with n vertices. Then*

$$d_2(v) = k(k-1).$$

In this paper, motivated by eccentric connectivity index, the authors introduced the leap eccentric connectivity index $L\xi^c(G)$ of a graph G which is defined as the sum of products of both the second degree and the eccentricity of every vertex in G. The exact values of $L\xi^c(G)$ for some well-known graphs are obtained. Some properties and bounds on leap eccentric connectivity index are found.

Definition 1.5. *For a graph $G = (V, E)$, the leap eccentric connectivity index of G is defined by*

$$L\xi^c(G) = \sum_{v \in V(G)} d_2(v)e(v),$$

where $d_2(v) = |\{u \in V(G) : d(u, v) = 2\}|$ and $e(v) = max\{d(u, v)$: for every $u \in V(G)\}$.

2 Leap eccentric connectivity index of some standard graphs

In this section, we present the exact values of leap eccentric connectivity indices of some graph classes.

Proposition 2.1. *The leap eccentric connectivity index of a complete bipartite graph $K_{r,s}$ is given by*

$$L\xi^c(K_{r,s}) = 2r(r-1) + 2s(s-1).$$

Proof. Let $K_{r,s}$ be a complete bipartite graph with partite sets $V_r = \{v_1, v_2, \cdots, v_r\}$ and $V_s = \{u_1, u_2, \cdots, u_s\}$. Then, the eccentricity of a vertex of $K_{r,s}$ is given by $e(v/K_{r,s}) = 2$ for every $v \in V(K_{r,s})$ and also $d_2(v_i/K_{r,s}) = r - 1$ for $v_i \in V_r$ and $d_2(u_i/K_{r,s}) = s - 1$ for $u_j \in V_s$ where $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, s$. Thus, the leap eccentric connectivity index of $K_{r,s}$ is given by

$$\begin{aligned} L\xi^c(K_{r,s}) &= \sum_{v \in V_r} d_2(v/K_{r,s}) e(v/K_{r,s}) + \sum_{u \in V_s} d_2(u/K_{r,s}) e(u/K_{r,s}) \\ &= \sum_{i=1}^{r} (r-1) \cdot 2 + \sum_{j=1}^{s} (s-1) \cdot 2 \\ &= 2r(r-1) + 2s(s-1). \end{aligned}$$

□

Corollary 2.2. *For the star graph $K_{1,n-1}$ with $n \geq 1$ vertices, the leap eccentric connectivity index is given by*

$$L\xi^c(K_{1,n-1}) = 2(n-1)(n-2).$$

Theorem 2.3. *The leap eccentric connectivity index of a cycle C_n for $n \geq 3$ is given by*

$$L\xi^c(C_n) = \begin{cases} 0, & \text{if } n = 3, \\ 8, & \text{if } n = 4, \\ n^2, & \text{if } n \text{ is even, } n \neq 4, \\ n(n-1), & \text{if } n \text{ is odd, } n \neq 3. \end{cases}$$

Proof. For a cycle C_n with $n \geq 3$ vertices, we have

$$e(v/C_n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Also, for the second degrees of the vertices of C_n, we have

$$d_2(v/C_n) = \begin{cases} 0, & \text{if n= 3,} \\ 1, & \text{if n = 4,} \\ 2, & \text{otherwise.} \end{cases}$$

Hence, to compute the leap eccentric connectivity index of C_n, we consider the following cases:

Case 1: If $n = 3$, then $L\xi^c(C_n) = 0$.

Case 2: If $n = 4$, then $e(v/C_n) = 2$, and $d_2(v/C_n) = 1$ for each $v \in V(C_n)$. Thus,

$$\begin{aligned} L\xi^c(C_n) &= \sum_{v \in V(C_n)} d_2(v/C_n)e(v/C_n) \\ &= \sum_{v \in V(C_n)} 2 \cdot 1 = 2 \cdot 4 = 8. \end{aligned}$$

Case 3: If $n \geq 5$, we have the following subcases:

Subcase 3.1: If n is even, then

$$\begin{aligned} L\xi^c(C_n) &= \sum_{v \in V(C_n)} d_2(v/C_n)e(v/C_n) \\ &= \sum_{v \in V(C_n)} 2 \cdot \frac{n}{2} = n^2. \end{aligned}$$

Subcase 3.2: if n is odd, then

$$\begin{aligned} L\xi^c(C_n) &= \sum_{v \in V(C_n)} d_2(v/C_n)e(v/C_n) \\ &= \sum_{v \in V(C_n)} 2 \cdot \frac{n-1}{2} \\ &= n(n-1). \end{aligned}$$

Therefore,

$$L\xi^c(C_n) = \begin{cases} 0, & \text{if } n = 3, \\ 8, & \text{if } n = 4, \\ n^2, & \text{if } n \neq 4 \text{ is even}, \\ n(n-1), & \text{if } n \neq 3 \text{ is odd}. \end{cases}$$

□

Theorem 2.4. *For the path P_n with $n \geq 3$ vertices, the leap eccentric connectivity index is given by*

$$L\xi^c(P_n) = \begin{cases} \frac{3n^2-10n+12}{2}, & \textit{if } n \textit{ is even}, \\ \frac{3n^2-10n+11}{2}, & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. Let P_n be a path with $n \geq 3$ vertices and let $v_1, v_2, \cdots, v_n$ be the vertices of P_n. Then we have the following cases:

Case 1: If n is even, then we have

$$\begin{aligned} e(v_1) &= e(v_n) = n-1 \\ e(v_2) &= e(v_{n-1}) = n-2 \\ &\cdots\cdots \\ e(v_i) &= e(v_{n-i+1}) = n-i+1 \\ &\cdots\cdots \\ e(v_{\frac{n}{2}}) &= e(v_{n-\frac{n}{2}-1}) = \frac{n}{2}. \end{aligned}$$

Also, we have $d_2(v_1) = d_2(v_2) = d_2(v_{n-1}) = d_2(v_n) = 1$, and $d_2(v_i) = 2$, for $3 \leq i \leq n-2$. Thus,

$$\begin{aligned} L\xi^c(P_n) &= \sum_{i=1}^{n} d_2(v_i/P_n)e(v_i/P_n) \\ &= 1(n-1) + 1(n-2) + 2(n-3) + \cdots + 2\left(\frac{n}{2}\right) + 2\left(\frac{n}{2}\right) + \cdots + 2(n-3) \\ &+ 1(n-2) + 1(n-1) \\ &= 4\left[(n-1) + (n-2) + \cdots + \left(n - \frac{n}{2}\right)\right] - 2(n-1) - 2(n-2) \\ &= 4\left[\frac{n^2}{2} - \frac{n(n+2)}{8}\right] - 4n + 6 \\ &= 2n^2 - \frac{n(n+2)}{2} - 4n + 6 \\ &= \frac{4n^2 - n^2 - 2n - 8n + 12}{2} \\ &= \frac{3n^2 - 10n + 12}{2}. \end{aligned}$$

Therefore, $L\xi^c(P_n) = \frac{3n^2-10n+12}{2}$.

Case 2: If n is odd, then we have,

$$\begin{aligned} e(v_1) &= e(v_n) = n-1 \\ e(v_2) &= e(v_{n-1}) = n-2 \\ &\cdots\cdots \\ e(v_i) &= e(v_{n-i+1}) = n-i \\ &\cdots\cdots \\ e(v_{\frac{n-1}{2}}) &= e(v_{\frac{n+1}{2}}) = \frac{n+1}{2} = n - \left(\frac{n-1}{2}\right) \\ e(v_{\frac{n+1}{2}}) &= \frac{n-1}{2} = n - \left(\frac{n+1}{2}\right). \end{aligned}$$

Also we have $d_2(v_1) = d_2(v_2) = d_2(v_{n-1}) = d_2(v_n) = 1$ and $d_2(v_i) = 2$ for every $3 \leq i \leq n-2$. Thus

$$\begin{aligned}
L\xi^c(P_n) &= \sum_{i=1}^{n} d_2(v_i/P_n)e(v_i/P_n) \\
&= 1(n-1) + 1(n-2) + 2(n-3)+, ..., +2\left(n - \frac{n-1}{2}\right) + 2\left(n - \frac{n+1}{2}\right) \\
&+ 2\left(n - \frac{n-1}{2}\right)+, ..., +2(n-3) + 1(n-2) + 1(n-1) \\
&= 4\left[\frac{n(n-1)}{2} - \left(1 + 2 + 3+, ..., +\frac{n-1}{2}\right)\right] + 2\left(\frac{n-1}{2}\right) - 2(n-1) - 2(n-2) \\
&= 4\left[\frac{n(n-1)}{2} - \frac{\frac{n-1}{2}\left((\frac{n-1}{2}) + 1\right)}{2}\right] + n - 1 - 2n + 2 - 2n + 4 \\
&= 2n(n-1) - \frac{(n-1)(n+1)}{2} - 3n + 5 \\
&= \frac{3n^2 - 10n + 11}{2}
\end{aligned}$$

Therefore $L\xi^c(P_n) = \frac{3n^2-10n+11}{2}$.

□

Proposition 2.5. *For the complete graph K_n with $n \geq 4$ vertices, the leap eccentric connectivity index is given by*

$$L\xi^c(K_n) = 0.$$

Proposition 2.6. *For the wheel graph $W_{1,n-1}$ with $n \geq 4$ vertices, the leap eccentric connectivity index is*

$$L\xi^c(W_{1,n-1}) = 2(n-1)(n-4).$$

3 Bounds on leap eccentric connectivity index

Theorem 3.1. *For any r-regular graph G with n vertices, we have*

$$L\xi^c(G) \leq r(r-1)\theta(G) \tag{3.1}$$

and equality holds if and only if G is a $(C_3, C_4)-$ free graph.

Proof. Let G be an r-regular graph with n vertices. Then by Corollary 1.4, we obtain

$$\begin{aligned} L\xi^c(G) &= \sum_{v\in V(G)} d_2(v)e(v) \\ &\leq \sum_{v\in V(G)} r(r-1)e(v) \\ &= r(r-1)\sum_{v\in r(G)} e(v) \\ &= r(r-1)\theta(G). \end{aligned}$$

Again by Corollary 1.4, we have that $d_2(v) = r(r-1)$, if and only if G is (C_3, C_4)-free graph. This implies that the equality in Eqn. 3.1 holds if and only if G is a $(C_3, C_4)-$free graph. □

Theorem 3.2. *Let G be a connected graph with n vertices and m edges. Then*

$$L\xi^c(G) \leq diam(G)(M_1(G) - 2m).$$

Equality holds if and only if G is a self-centred (C_3, C_4)-free graph.

Proof. Let G be a connected graph with n vertices and m edges. Since $e(v) \leq diam(G)$ for every $v \in V(G)$, by Lemma 1.1, we obtain that $d_2(v) \leq \sum_{u\in N(V)} d(u) - d(v)$. Thus

$$\begin{aligned} L\xi^c(G) &= \sum_{v\in G} d_2(v)e(v) \\ &\leq \sum_{v\in G} d_2(v)diam(G) \qquad (5) \\ &= diam(G)\sum_{v\in G} d_2(v) \\ &\leq diam(G)\sum_{v\in G}\left(\sum_{u\in N(v)} d(u) - d(v)\right) \\ &\leq diam(G)\Big(M_1(G) - 2m\Big). \qquad (6). \end{aligned}$$

Equality holds in (5) if and only if G is a self-centred graph and by Lemma 1.1, equality holds in (6) if and only if G is a (C_3, C_4)-free graph. □

Theorem 3.3. *For any graph G, we have*

$$L\xi^c(G) \leq (n+1)\theta(G) - \xi^c(G) - E_1(G).$$

The bound is attained for C_5 and C_6.

Proof. Let G be a connected graph with n vertices. Then by Lemma 1.2, we get

$$\begin{aligned} L\xi^c(G) &= \sum d_2(v)e(v) \\ &\leq \sum \big[(n+1) - d_1(v) - e(v)\big]e(v) \\ &\leq \sum (n+1)e(v) - d_1(v)e(v) - e^2(v) \\ &\leq (n+1)\sum e(v) - \sum d_1(v)e(v) - \sum e^2(v). \end{aligned}$$

Therefore $L\xi^c(G) \leq (n+1)\theta(G) - \xi^C(G) - E_1(G)$. □

Theorem 3.4. *Let G be a graph with n vertices. Then*

$$L\xi^c(G) \leq \Big(LM_1(G)E_1(G)\Big)^{\frac{1}{2}}. \tag{7}$$

Equality holds if and only if G is an r-regular self-centered graph.

Proof. Let G be a graph with n vertices. Then by using Cauchy-Schwarz inequality, we get

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Setting $a_i = d_2(v_i)$ and $b_i = e(v_i)$, we obtain

$$\left(\sum_{v\in V(G)} d_2(v)e(v)\right)^2 \leq \left(\sum_{v\in V(G)} d_2^2(v)\right)\left(\sum_{v\in V(G)} e^2(v)\right).$$

Therefore $L\xi^c(G) \leq \Big(LM_1(G)E_1(G)\Big)^{\frac{1}{2}}$. Equality holds in Cauchy-Schwarz inequality if and only if $a_1 = a_2 = \cdots = a_n$ and $b_1 = b_2 = \cdots = b_n$ or a_i or $b_i = 0$ for $i = 1, 2, \cdots, n$. Hence equality holds in (7) if and only if $d_2(v_1) = d_2(v_2) = \cdots = d_2(v_1)$ and $e(v_1) = e(v_2) = \cdots = e(v_n)$ or $d_2(v) = 0$ for every $v \in V(G)$. That means equality holds in (7) if and only if G is a regular self-centered graph. □

Theorem 3.5. *Let G be a graph with n vertices and let $rad(G) = r(G)$. Then*

$$L\xi^c(G) \leq r(G)[n(n+1) - 2m - nr(G)].$$

Proof. Let G be a graph with n vertices and m edges. Since $rad(G) \leq e(v) \leq diam(G)$ for every $v \in V(G)$, by Lemma 1.2, we have $d_2(v) \leq n+1-d_1(v)-e(v)$. Then

$$\begin{aligned} L\xi^c(G) &= \sum d_2(v)e(v), \\ &\leq \sum \big(n+1-d_1(v)-r(G)\big)r(G) \\ &\leq r(G)[n(n+1) - 2m - nr(G)]. \end{aligned}$$

□

Since $n-1 \leq m$ for every connected graph with n vertices and m edges, the following result immediately follows by Theorem 3.6:

Corollary 3.6. *Let G be a connected graph with n vertices and m edges. Then*

$$L\xi^c(G) \leq rad(G)[n^2 - n(1 + rad(G)) + 2].$$

Theorem 3.7. *Let G be a (C_3, C_4)-free graph. Then*

$$L\xi^c(G) \geq \frac{2m}{n}\bigg(2m - n\bigg).$$

Equality holds if and only if $G = K_n$.

Proof. Let G be a $(C_3, C_4)-$free graph. Then $d_2(v) = \sum_{v \in V(G)} d(u) - d(v)$. Since

$$\begin{aligned} L\xi^c(G) &= \sum d_2(v)e(v) \geq \sum d_2(v) \\ &\leq \sum \bigg(\sum d(u) - d(v)\bigg) \\ &= M_1(G) - 2m, \end{aligned}$$

as $M_1(G) \geq \frac{4m^2}{2}$, we get $L\xi^c(G) \geq \frac{4m^2}{n} - 2m$. Therefore, $L\xi^c(G) \geq \frac{2m}{n}\bigg(2m-n\bigg)$. Equality holds if and only if $e(v) = 1$ for every $v \in V(G)$ and G is an r-regular graph. Therefore $G = K_n$. □

Theorem 3.8. *Let G be a graph with n vertices, m edges and $rad(G) = r(G)$. Then*

$$L\xi^c(G) \geq \frac{2m\, r(G)(2m - n)}{n}.$$

The bound is attained for C_n, $n \geq 3$.

Proof. Chebyshev's inequality state that for any non-increasing sequences $a_1 \geq a_2 \geq \cdots \geq a_n$, and $b_1 \geq b_2 \geq \cdots \geq b_n$, we have

$$n \sum a_i b_i \geq \sum a_i \sum b_i.$$

Now, by setting $a_i = d_2(v_i)$ and $b_i = e(v_i)$ for $i = 1, 2, \cdots, n$, by Lemma 1.1, we obtain

$$n \sum d_2(v_i)e(v_i) \geq \sum d_2(v_i) \sum e(v_i),$$

and hence

$$\begin{aligned} nL\xi^c(G) &\geq (M_1(G) - 2m)nr(G) \\ &\geq \bigg(\frac{4m^2}{n} - 2m\bigg)nr(G) \\ &\geq \bigg(\frac{4m^2 - 2mn}{n}\bigg)nr(G) \\ &= 2m(2m - n)r(G). \end{aligned}$$

Therefore, $L\xi^c(G) \geq \frac{2mr(G)(2m-n)}{n}$. □

Theorem 3.9. *Let G be a connected graph with n vertices and let n_0 be the number of vertices with $e(v) = 1$ in G. Then*

$$L\xi^c(G) \geq 2(n - n_0)$$

with equality holding if and only if G is an $(n-2)-$regular $self-$centred graph with $diam(G) = 2$ such that n is even.

Proof. Let $V_0 = \{v \in V(G); e(v) = 1\}$. Then $d_2(v) = 0$ for every $v \in V_0$ and for every $u \in V - V_0$, we have $e(u) \geq 2$, and $d_2(u) \geq 1$. Hence

$$\begin{aligned} L\xi^c(G) &= \sum_{v \in V_0} d_2(v)e(v) + \sum_{v \notin V_0} d_2(v)e(v) \\ &\geq \sum_{v \in V_0} 0.1 + \sum_{v \notin V_0} 1.2 \\ &= 0 + 2|V(G) - V_0| \\ &= 2(n - n_0). \end{aligned}$$

The equality holds if and only if $e(v) = 2$ and $d_2(v) = 1$ for every $v \in V(G)$ that means $d_1(v) = n - 2$ and $diam(G) = e(v) = 2$. Therefore G is $(n - 2)-$regular self centered graph with $diam(G) = 2$ and n is even. □

Theorem 3.10. *Let G be a connected graph with $n \geq 3$ vertices and m edges. Then*

$$L\xi^c(G) \leq nM_1(G) - LM_3(G) - 2nm.$$

The bound is attained for P_4.

Proof. Let G be a connected graph with n vertices and m edges. Since $e(v) \leq n - d(v)$ for every $v \in V(G)$, it follows that

$$\begin{aligned} L\xi^c(G) &= \sum_{v \in V(G)} d_2(v)e(v) \leq \sum_{v \in V(G)} d_2(v)(n - d_1(v)) \qquad (7) \\ &= \sum_{v \in V(G)} nd_2(v) - \sum d_1(v)d_2(v) \\ &= n \sum_{v \in V(G)} d_2(v) - \sum d_1(v)d_2(v) \end{aligned}$$

and by Lemma 1.1, using the definition of $LM_3(G)$, we get

$$L\xi^c(G) \leq n\Big(M_1(G) - 2m\Big) - LM_3(G) \qquad (8)$$ □

Corollary 3.11. *Let G be a connected graph with n vertices and m edges. Then*

$$L\xi^c(G) \leq (n - \delta)[M_1(G) - 2m].$$

Theorem 3.12. *Let G be a connected graph with n vertices and m edges. Then*

$$L\xi^c(G) \leq n^2(n-1) - 2m(2n-1) + M_1(G).$$

Lemma 3.13. *(Diaz-Metcalf inequality) Let the real numbers $a_i \neq 0$ and b_i for $i = 1, 2, 3, \cdots, n$, satisfy $m \leq \frac{b_i}{a_i} \leq M$. Then*

$$\sum_{i=1}^{n} b_i^2 + mM \sum_{i=1}^{n} a_i^2 \leq (M+m) \sum_{i=1}^{n} a_i b_i.$$

Equality holds if and only if $b_i = ma_i$ or $b_i = Ma_i$.

Theorem 3.14. *Let G be a graph with n vertices with $\Delta \leq n-2$ and $\delta \geq 1$. Then*

$$L\xi^c(G) \geq \frac{rad(G)diam(G)}{rad(G) + diam(n-2)} \left[LM_1(G) + \frac{n-2}{rad(G)diam(G)} E_1(G) \right].$$

Proof. By setting $a_i = e(v_i)$ and $b_i = d_2(v_i)$ in Diaz-Metcalf inequality, we get

$$\sum d_2^2(v) + mM \sum e^2(v) \leq (m+M) \sum d_2(v)e(v).$$

Since $1 \leq d_2(v) \leq n-2$ and $rad(G) \leq e(v) \leq diam(G)$, for every $v \in V(G)$, we obtain

$$\frac{1}{diam(G)} \leq \frac{d_2(v)}{e(v)} \leq \frac{n-2}{rad(G)},$$

i.e, $m = \frac{1}{diam(G)}$ and $M = \frac{n-2}{rad(G)}$. Hence,

$$LM_1(G) + \frac{n-2}{rad(G)} E_1(G) \leq \left(\frac{1}{diam(G)} + \frac{n-2}{rad(G)} \right) L\xi^c(G).$$

Therefore,

$$L\xi^c(G) \geq \frac{rad(G)diam(G)}{rad(G) + diam(G)(n-2)} \left[LM_1(G) + \frac{(n-2)}{rad(G)diam(G)} E_1(G) \right].$$

Equality holds if and only if for every $v \in V(G)$, $d_2(v) = \frac{1}{diam(G)} e(v)$ or $d_2(v) = \frac{1}{rad(G)} e(v)$ if and only if $e(v) = rad(G) = diam(G)$ if and only if $d_2(v) = 1$ if and only if $d_1(v) = n-2$ and G is a $(n-2)-$regular self-centered graph. □

Lemma 3.15. *[21] Let a_i and b_i, $1 \leq i \leq n$ be positive real numbers. Then*

$$\left| n \sum_{i}^{n} a_i b_i - \sum_{i}^{n} a_i \sum_{i}^{n} b_i \right| \leq \alpha(n)(A-a)(B-b),$$

where a, b, A and B are real constants such that for each i, $1 \leq i \leq n$, we have $a \leq a_i \leq A$ and $b \leq b_i \leq B$. Further $\alpha(n) = n \left\lceil \frac{n}{2} \right\rceil \left(1 - \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil \right)$.

Theorem 3.16. *Let G be a graph with n vertices and m edges. Then*

$$L\xi^c(G) \leq \frac{1}{n}\Bigg[(M_1(G)-2m)\theta(G)+\alpha(n)(n-2)(diam(G)-rad(G))\Bigg].$$

with equality holds if and only if G is a $(C_3, C_4)-$free self-centered graph.

Proof. Let $a_i = d_2(v)$, $b_i = e(v)$, $a = 0$, $A = n-2$, $b = rad(G)$ and $B = diam(G)$. By Lemma 3.15, we have $0 \leq d_2(v)$ and $rad(G) \leq e(v) \leq diam(G)$ for every $v \in V(G)$. Then we get

$$\left|n\sum_{v\in V(G)} d_2(v)e(v) - \sum_{v\in V(G)} d_2(v)\sum_{v\in V(G)} e(v)\right| \leq \alpha(n)(n-2-0)\Big(diam(G)-rad(G)\Big).$$

By the definition of $L\xi^c(G)$ and $\theta(G)$, we obtain

$$nL\xi^c(G) \leq \theta(G)\sum d_2(v) + \alpha(n)(n-2)\Big(diam(G)-rad(G)\Big).$$

By Lemma 1.1, we obtain

$$nL\xi^c(G) \leq \theta(G)(M_1(G)-2m) + \alpha(n)(n-2)\Big(diam(G)-rad(G)\Big).$$

Equality holds if and only if G is a $(C_3, C_4)-$free self-centered graph. □

4 Leap eccentric connectivity index of join graphs

Let G and H be connected graphs with n_1, n_2, m_1, and m_2, denote the numbers of vertices and edges of these two graphs, respectively, and let $A_G = \{v \in V(G), e(v) = 1\}$ and $A_H = \{v \in V(H), e(v) = 1\}$. Then

Theorem 4.1. *Let G and H be two connected graphs. Then*

$$L\xi^c(G+H) = 2\big[n_1^2-(n_1-3)|A_G|-n_1-2m_1\big]+2\big[n_2^2-(n_2-3)|A_H|-n_2-2m_2\big].$$

Proof. Let G and H be connected graphs and let $A_G = \{v \in V(G), e(v) = 1\}$ and $A_H = \{v \in V(H), e(v) = 1\}$. Then for every $v \in A_G$, $d_2(v/G+H) = 0$ and $e(v/G+H) = 2$; also for every $v \in V(G) - A_G$, $d_2(v/G+H) = n_1 - 1 - d_1(v)$ and $e(v/G+H) = 2$. Similarly, for every $v \in A_H$, $d_2(v/G+H) = 0$ and $e(v/G+H) = 2$; also for every $v \in V(H) - A_H$, $d_2(v/G+H) = n_2 - 1 - d_1(v/H)$

and $e(v/G+H)=2$. Hence

$$\begin{aligned}
Ł\xi^c(G+H) &= \sum_{v\in V(G+H)} d_2(v/G+H)e(v/G+H)\\
&= \sum_{v\in V(G)} d_2(v/G+H)e(v/G+H) + \sum_{v\in V(H)} d_2(v/G+H)e(v/G+H)\\
&= \sum_{v\in A_G} d_2(v/G+H)e(v/G+H) + \sum_{v\in V(G)-A_G} d_2(v/G+H)e(v/G+H)\\
&+ \sum_{v\in A_H} d_2(v/G+H)e(v/G+H) + \sum_{v\in V(H)-A_H} d_2(v/G+H)e(v/G+H)\\
&= 0 + \sum_{v\in A_G} [n_1-1-d_1(v)]\cdot 2 + 0 + \sum_{v\in A_H} [n_2-1-d_1(v/H)]\cdot 2\\
&= 2\big[|v(G)-A_G|(n_1-1) - \sum_{v\in V(G)-A_G} d(v)\big]\\
&+ 2\big[|v(H)-A_H|(n_2-1) - \sum_{v\in V(H)-A_H} d(v/H)\big]\\
&= 2\big[n_1^2-(n_1-3)|A_G|-n_1-2m_1\big] + 2\big[n_2^2-(n_2-3)|A_H|-n_2-2m_2\big].
\end{aligned}$$

□

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