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Research Article

Constant-Ratio Blended Models of Curves and Surfaces in Euclidean 3-Space

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We consider a simple blending method for parametric curves and surfaces. The described blending method offers a systematic approach to creating smooth transitions between parametric curves and surfaces while maintaining C^n-continuity. It utilizes the parametrizations of the original objects to achieve predictable variations in shape and enables the generation of families of blending curves and surfaces with controlled characteristics. Therefore, considering the blending of the curves/surfaces to be constant-ratio objects, some results on the blending functions are obtained.

1. Introduction

Blending curves and surfaces is a fundamental operation in geometric modeling that enables the creation of smooth transitions and connections between different geometrical entities. This task is essential across various disciplines including computer-aided design (CAD), computer graphics, animation, and industrial design.

The method which is mentioned from [1] that introduces a simple blending method for parametric curves and surfaces producing families of parametrically defined C^n-continuous blending curves and surfaces seems very interesting.

The flexibility of this method seems to depend on the parameterization of the curves or surfaces being blended, allowing for various shapes and transformations.

Explicit blending, implicit blending, and parametric blending seem to offer different approaches to achieve blending in curves and surfaces, each with its own advantages and use cases [1–4]. It is great to see how these techniques can be applied and demonstrated through examples.

Blending surfaces in the CAD/CAM industry refer to secondary surfaces that are utilized to smoothly connect primary functional surfaces. These secondary surfaces assist in creating smooth transitions between intersecting surfaces or joining disconnected surfaces seamlessly [5]. Various methods and research have been conducted on parametric blending surfaces to achieve these smooth transitions effectively [6–17].

Geometrical modeling indeed encompasses various techniques in three-dimensional CAD. There are three main types of geometrical modeling used, namely, line or wireframe, surface, and solid modeling.

Solid modeling is the most advanced form, where not only the surfaces but also the volumes (solids) of objects are represented digitally. Solid models define the complete geometry of an object, including its interior space, which makes them ideal for engineering and manufacturing applications [18].

Computer-based systems for modeling the geometry of rigid solid objects have gained significant importance across various disciplines such as mechanical and civil engineering, architecture, computer graphics, and computer vision. These systems enable precise representation and manipulation of three-dimensional solid geometries, which is crucial in industries involved in manufacturing and construction.

In scientific and engineering fields, three-dimensional solid geometry serves as a foundational tool for designing and analyzing complex structures and systems. It allows engineers and designers to simulate and optimize the

performance of mechanical components, buildings, and infrastructure. In architecture, these models facilitate the visualization and planning of structures, aiding in the creation of aesthetically pleasing and functional designs [19].

Blend surfaces in solid modeling systems demand their explicit definitions, capable of supplying every detail needed in further processing.

The formulation of blend surfaces typically draws upon principles from differential geometry. This mathematical framework allows for the precise derivation of blend equations that govern the shape and characteristics of these surfaces. By employing differential geometry, researchers and developers can analyze and solve blend equations for various types of surfaces that need to be seamlessly integrated or transitioned in solid modeling [20].

Universal blending methods for curves and surfaces are techniques designed to smoothly transition between or combine any types of curves and surfaces, regardless of their initial shape or structure. These methods aim to provide flexibility, control, and smoothness in geometric modeling across a variety of applications, from industrial design to animation. There are several universal blending methods for curves and surfaces: Bezier blending, B-spline, NURBS, Hermite interpolation, rolling ball method, subdivision surfaces, and polynomial blending. Each method has its specific advantages and disadvantages depending on the level of control, smoothness, and complexity required for the task. For high precision and flexibility, NURBS, B-splines and subdivision surfaces are often the methods of choice, while simpler tasks can often rely on Bezier, Hermite, or polynomial blending methods.

Indeed, blending constant ratio curves and surfaces is crucial in many fields, including engineering, manufacturing, robotics, computer graphics, and biomedical design. In engineering design—particularly in CAD, structural optimization, or fluid interface modeling—the blending of curves and surfaces plays a critical role. This technique ensures smooth transitions between shapes, improving esthetic appeal, structural integrity, aerodynamic efficiency, and motion control. Blending curves and surfaces with a constant ratio create seamless and visually pleasing designs. The perfection of design shape is beneficial for modeling. This method prevents sharp edges and sudden changes in design shapes. This allows for a natural flow between shapes and objects, eliminates visual weaknesses in the design, and creates shapes that allow for smoother and more precise movement.

In this study, the Euclidean metric is used in curves and surface blending and this metric is common due to its simplicity and analytical convenience. So, it is standard practice to analyze this in the Euclidean metric, but it does not always reflect the most meaningful or perceptually relevant properties of shapes. It fails to capture the intrinsic, perceptually relevant, or nontransformation-dependent aspects of shapes, and more complex metrics can provide deeper, more robust insights into shape similarity and deformation. So, it can be use the further studies the alternative metric such as elastic metric, TPS metric, and geodesic distance.

The rest of this paper is organized as follows. In Section 2, we review mathematical preliminaries on the constant-ratio curves and surfaces and the blending of two parametric curves and surfaces. In Section 3, we obtain some results on blending curves and surfaces to be constant ratio.

2. Materials and Methods

2.1. Constant-Ratio Curves. In this subsection, we will consider constant-ratio curves in Euclidean spaces introduced in [21]. The definition of constant-ratio parametric curves can be revised from this reference. Also, see [22] for the characterization of constant-ratio curves and surfaces in low dimension.

Before moving on to the definition, the importance of constant-ratio functions should be emphasized. Curves with a constant ratio tend to show uniform curvature.

The fact that the ratio between the position vector and the velocity (or tangent) vector is constant indicates that there is regularity in geometric growth. In differential geometry and physics, such functions describe motion or growth patterns with constant relative acceleration or scale-invariant dynamics.

By adjusting the ratio, designers or analysts can control the curvature change in geometric modeling, which can lead to better control in optimization, design, or physics simulations. As a result, constant-ratio functions provide a bridge between geometric regularity and analytic control.

Definition 1. Let $\gamma: I \subset \mathbb{R}^n$ be a unit speed curve in $\mathbb{R}^n$. Then, the position vector γ of the curve can be decomposed into its tangential and normal components at each point as follows:

$$\gamma = \gamma^T + \gamma^N. \tag{1}$$

If the ratio $\|\gamma^T\|: \|\gamma^N\|$ is constant on $\gamma(I)$, then γ is said to be of constant ratio or equivalently $\|\gamma^T\|: \|\gamma\| = c = \text{const}$.

For a unit speed regular curve γ in $\mathbb{R}^n$, the gradient of the distance function $\rho = \|\gamma(s)\|$ is obtained by

$$\text{grad}\rho = \frac{d\rho}{ds}\gamma'(s) = \frac{\langle \gamma(s), \gamma'(s)\rangle}{\|\gamma(s)\|}\gamma'(s), \tag{2}$$

where $\gamma'(s)$ is the tangent vector of γ.

The following results characterize the constant-ratio curves [23].

Theorem 1. *Let $\gamma: I \subset \mathbb{R}^n$ be a unit speed curve in $\mathbb{R}^n$. Then, γ is of constant ratio with $\|\gamma^T\|: \|\gamma\| = c$ if and only if $\|grad\rho\| = c$, $c \leq 1$ which is constant.*

Theorem 2. *Let $\gamma: I \subset \mathbb{R}^n$ be a unit speed curve in $\mathbb{R}^n$. Then, $\|grad\rho\| = c$ holds for a constant c if and only if one of the following three cases occurs:*

i. $\gamma(I)$ is contained in a hypersphere centered at the origin.

ii. $\gamma(I)$ is an open portion of a line through the origin.

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iii. $\gamma(s) = cs\alpha(u)$, $c \in (0,1)$, *where* $\alpha = \alpha(u)$ *is a unit speed curve on the unit sphere of* $\mathbb{R}^n$ *centered at the origin and* $u = (\sqrt{1-c^2}/c)\ln s$.

The following result derives from Lemma 5.5 given in [23].

Corollary 1. *Let* γ: $I \subset \mathbb{R}^n$ *be a unit speed curve in* $\mathbb{R}^n$. *Then, up to a translation of the arc length function s, we have*

i. $\|grad\rho\| = 0 \Longleftrightarrow \gamma(I)$ *is contained in a hypersphere centered at the origin.*

ii. $\|grad\rho\| = 1 \Longleftrightarrow \gamma(I)$ *is an open portion of a line through the origin.*

iii. $\|grad\rho\| = c \Longleftrightarrow \gamma(I)$, $\rho = \|\gamma(s)\| = cs$, *for* $c \in (0,1)$. *For* $n = 2$, *the curvature of* γ *is*

$$\kappa(s) = \pm \frac{\sqrt{1-c^2}}{c\sqrt{s^2+b}}, \tag{3}$$

for some constant b.

2.2. Blending of Two Parametric Curves. A C^n–continuous curve γ can be described parametrically using functions that relate a parameter (often denoted as t) to coordinates in space.

The derivatives of the parametric functions with respect to the parameter t are crucial. These derivatives give information about the tangent direction, curvature, and other geometric properties of the curve.

The definition of C^n-continuity of parametric curves can be revised in [24].

Definition 2. Two regular C^n-continuous curves $\gamma_1 = \gamma_1(t)$ and $\gamma_2 = \gamma_2(t)$ have C^n-contact at a common point p, if the derivatives $\gamma_1^{(k)}$ and $\gamma_2^{(k)}$, $k = 0, 1, \ldots, n$ at point p are the same. Furthermore, two regular C^n-continuous curves $\gamma_1 = \gamma_1(t)$ and $\gamma_2 = \gamma_2(t)$, have G^n-contact at a common point p if they have C^n-contact at point p with respect to some local reparametrizations of these two curves.

Definition 3. A regular C^n-continuous curve β that has G^n-contact to two regular C^n-continuous curves γ_1 and γ_2 at points $p_1 \in \gamma_1$ and $p_2 \in \gamma_2$, respectively, is called a G^n-blending curve of γ_1 and γ_2 (Figure 1).

The following result is due to [1].

Theorem 3. *Let* $\gamma_1 = \gamma_1(t)$ *and* $\gamma_2 = \gamma_2(t)$, $t \in [0,1]$, *be two regular* C^n*-continuous plane or space curves (base curves) and let* $f(t)$, $t \in [0,1]$, *be a* C^n*-continuous real function (blending function) with the following properties:*

$$\begin{aligned} f(0) &= 1, \\ f(1) &= 0, \\ f^{(k)}(0) &= f^{(k)}(1) = 0, \quad \text{for } k = 1, \ldots, n, \end{aligned} \tag{4}$$

then the blending curve

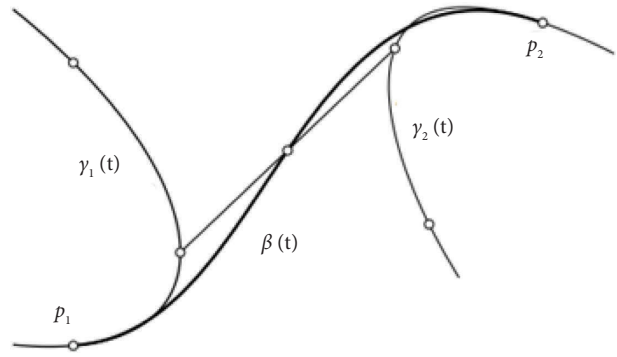

Figure 1: G^n–blending curve.

$$\beta(t) = f(t)\gamma_1(t) + (1 - f(t))\gamma_2(t), \quad t \in [0,1] \tag{5}$$

has C^n*–contact to the curves* γ_1 *and* γ_2 *at the points* $\gamma_1(0) = p_1$ *and* $\gamma_2(1) = p_2$, *i.e.,*

$$\begin{aligned} \beta^{(k)}(0) &= \gamma_1{}^{(k)}(0), \\ \beta^{(k)}(1) &= \gamma_2{}^{(k)}(1), \\ k &= 0, \ldots, n. \end{aligned} \tag{6}$$

Definition 4 (Rational G^n-blending functions). The rational functions

$$c_{n,\mu}(t) = \frac{\mu(1-t)^{n+1}}{\mu(1-t)^{n+1} + (1-\mu)t^{n+1}}, \quad t \in [0,1] \tag{7}$$

are called G^n-blending functions. Here $0 < \mu < 1$ is called design parameter for any $n \geq 0$ (see [1]). For the case $\mu = 1/2$, one can get

$$c_{n,\mu}(t) = \frac{(1-t)^{n+1}}{(1-t)^{n+1} + t^{n+1}}. \tag{8}$$

Corollary 2. *Let* $\gamma_1 = \gamma_1(t)$ *and* $\gamma_2 = \gamma_2(t)$, $t \in [0,1]$, *be two regular* C^n*-continuous plane or space curves. Then, for any design parameter* μ, *the parametric curve*

$$\beta(t) = c_{n,\mu}(t)\,\gamma_1(t) + \left(1 - c_{n,\mu}(t)\right)\gamma_2(t) \tag{9}$$

has C^n*-contact with curve* γ_1 *at* $\gamma_1(0) = p_1$, *and* C^n*-contact with curve* γ_2 *at* $\gamma_2(1) = p_2$. *If* γ_1 *and* γ_2 *are rational, then* β *is too.*

Definition 5 (Thumb weight). For the nonlinear real-valued functions

$$\begin{aligned} p_1(t) &= \frac{t}{(1-\lambda+\lambda t)}, \\ p_2(t) &= 1 - p_1(1-t), \end{aligned} \tag{10}$$

$0 \leq \lambda < 1$ (see [5]).

Corollary 3. *Let* $\gamma_1 = \gamma_1(t)$ *and* $\gamma_2 = \gamma_2(t)$, $t \in [0,1]$, *be two regular* C^n*-continuous plane or space curves. Then, for any (design) parameter* λ, *the parametric curve*

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$$\beta(t) = f(t)\gamma_1(p_1(t)) + (1 - f(t))\gamma_2(p_2(t)), \quad t \in [0,1] \tag{11}$$

has C^n-contact with the curve γ_1 at $\gamma_1(0)$, and C^n-contact with the curve γ_2 at $\gamma_2(1)$. Here, $f(t)$ is blending function defined as in Theorem 3.

2.3. Constant-Ratio Surfaces. In this subsection, we will consider constant-ratio surfaces in Euclidean spaces introduced in [21]. The definition of constant-ratio parametric surfaces can be revised from this reference.

Let $M \subset \mathbb{R}^n$ be a local surface given by the regular patch $X(s,t)$. The gradient of the distance function $\rho = \|X\|$ of M, with the use of orthonormal frame $\{e_1, e_2\}$ of M is given by

$$\text{grad}\rho = \sum_{j=1}^{2} \frac{\langle X(s,t), e_j \rangle}{\|X(s,t)\|} e_j. \tag{12}$$

Furthermore, the norm of gradρ is obtained by

$$\|\text{grad}\rho\| = \sum_{j=1}^{2} \frac{\langle X(s,t), e_j \rangle}{\|X(s,t)\|}. \tag{13}$$

Definition 6. Let $M \subset \mathbb{R}^n$ be a local surface given by the regular patch $X(s,t)$. For the distance function ρ of M, if the condition

$$\|\text{grad}\rho\| = c, \quad c \le 1 \tag{14}$$

is satisfied, then M is called a constant-ratio surface.

The following result is derived from Theorem 1, which is given in [21].

Theorem 4. *Let $X{:}D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ be a regular patch of a local surface M in Euclidean n-space $\mathbb{R}^n$. If M is a constant-ratio surface, then*

i. $\|grad\rho\| = 0$ implies that M is an open portion of a sphere centered at the origin.

ii. $\|grad\rho\| = 1$ implies that M is an open portion of a cone vertex at the origin, where $\rho = \|X\|$ is the distance function on the surface M.

iii. $\|grad\rho\| = c$ implies that $c \in (0,1)$.

2.4. Blending of Two Parametric Surfaces. The definition of C^n continuity of parametric surfaces can be revised in [25].

Definition 7. Two regular C^n-continuous surfaces $X_1 = X_1(s,t)$ and $X_2 = X_2(s,t)$ have C^n-contact at a common point $p = X_1(s_1,t_1) = X_2(s_2,t_2)$ if all partial derivatives up to the nth order of X_1 and X_2 at point p are the same. In other words,

$$\left.\frac{\partial^k X_1}{\partial s^i \partial t^{k-i}}\right|_{(s_1,t_1)} = \left.\frac{\partial^k X_2}{\partial s^i \partial t^{k-i}}\right|_{(s_2,t_2)}, \tag{15}$$

$i = 0,1,\ldots,k; k = 1,\ldots,n$ (see [13]).

Definition 8. Two regular C^n-continuous surfaces X_1 and X_2 have G^n-contact at a common point p if they have C^n-contact at point p with respect to some local reparameterizations of the two surfaces (see [13, 26]).

Definition 9. A regular C^n-continuous surface X that has G^n-contact with two regular C^n-continuous surfaces X_1 and X_2 at curves $\gamma_1 \subset X_1$ and $\gamma_2 \subset X_2$, respectively, is called a G^n-blending surface of X_1 and X_2 [1].

The G^n-blending method introduced for curves can be extended to surfaces via the following theorem of [1].

Theorem 5. *Let $X_1 = X_1(s,t)$ and $X_2 = X_2(s,t)$, $s \in [s_1,s_2]$, $t \in [0,1]$, be regular C^n-continuous (base) surfaces, and let $f(t)$, $t \in [0,1]$ be a C^n-continuous real valued (blending) function with the following properties:*

$$\begin{aligned} f(0) &= 1, \\ f(1) &= 0, \\ f^{(k)}(0) &= f^{(k)}(1) = 0, \quad \text{for } k = 1,\ldots,n. \end{aligned} \tag{16}$$

Then, the surface patch (blending surface)

$$X(s,t) = f(t)X_1(s,t) + (1 - f(t))X_2(s,t), \tag{17}$$

$s \in [s_1,s_2], t \in [0,1]$, has C^n- contact to the surface patch X_1 along the s-parameter curve $\gamma_1 = X_1(s,0)$ and with X_2 along the s-parameter curve $\gamma_2 = X_2(s,1)$, $s \in [s_1,s_2]$.

Remark 1. (i) The curves $\gamma_1 = X_1(s,0)$ and $\gamma_2 = X_2(s,1)$, $s \in [s_1,s_2]$ are the contact curves between the base surfaces X_1 and X_2 and the blend surface X. The curves $X_1(s,1)$ and $X_2(s,0)$, $s \in [s_1,s_2]$, are called auxiliary curves. Contact curves and auxiliary curves are boundaries of the surface patches to be blended (see [21]). (ii) The curves $\beta(t) = X(s_0,t)$, $s_0 = \text{const.}$, are profile curves, which locally define the shape of the blend surface.

3. Results and Discussion

In shape blending or transformation, a widely used approach involves establishing correspondence between vertices of the source and target shapes. Studies on the interpolation process are mainly concentrated in deriving interpolation methods in order that a smooth transformation can be obtained without self-intersection in the in-between shapes [27].

3.1. Constant-Ratio Blended Models of Two Parametric Curves. Let $\gamma_1 = \gamma(s)$ be a unit speed, regular C^n–continuous curve, and $\gamma_2 = \gamma'(s)$ is the other base curve. Then, the blending curve β has the parametrization of the form

$$\beta(s) = f(s)\gamma(s) + (1 - f(s))\gamma'(s), \quad s \in [0,1]. \tag{18}$$

The distance function of the blending curve β becomes

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$$\tilde{\rho} = \|\beta(s)\| = \sqrt{f^2(s)\rho^2 + (1-f(s))^2 + 2f(s)(1-f(s))\langle\gamma(s),\gamma'(s)\rangle}, \tag{19}$$

where $\rho = \|\gamma(s)\|$, $\|\gamma'(s)\| = 1$. Furthermore, using

$$\text{grad}\rho = \frac{d\rho}{ds}\gamma'(s) = \frac{\langle\gamma(s),\gamma'(s)\rangle}{\|\gamma(s)\|}\gamma'(s), \tag{20}$$

we get

$$\left|\langle\gamma(s),\gamma'(s)\rangle\right| = \|\text{grad}\rho\|\rho = \rho(s)\frac{d\rho}{ds}. \tag{21}$$

Consequently, the distance function $\tilde{\rho}$ of blending curve β turns into

$$\tilde{\rho} = \sqrt{f^2(s)\rho^2 + (1-f(s))^2 + 2f(s)(1-f(s))\rho(s)\rho'(s)}. \tag{22}$$

Assume that the base curve γ is constant ratio of the form $\rho = \|\gamma(s)\| = cs$. Then,

$$\tilde{\rho} = \sqrt{f^2(s)c^2s^2 + (1-f(s))^2 + 2c^2f(s)(1-f(s))s} \tag{23}$$

holds. By the use of equation (2), the norm of the gradient of the distance function $\tilde{\rho}$ becomes $\|\text{grad}\tilde{\rho}\| = |(d\tilde{\rho}/ds)|\|\beta'(s)\|$. Hence, for the case $|(d\tilde{\rho}/ds)| = 0$, gradient of the distance function $\tilde{\rho}$ vanishes identically. Therefore, by Corollary 3, the blending curve β is contained in a hypersphere centered at the origin. Consequently, differentiating equation (23) and using the condition $|(d\tilde{\rho}/ds)| = 0$, we obtain the following differential equation:

$$(1-f(s))\left(c^2f(s) + \left(c^2s-1\right)f'(s)\right) + f(s)c^2s\left(f'(s)(s-1) + f(s)\right) = 0. \tag{24}$$

By the use of the Maple command, one can get the solution of the differential equation (24) as

$$f(s) = \frac{1 - c^2s + \sqrt{1 - 2c^2s + c^4s^2 - bc^2s^2 + 2c^2bs - b}}{c^2s^2 - 2c^2s + 1}, \tag{25}$$

or

$$f(s) = \frac{-1 + c^2s + \sqrt{1 - 2c^2s + c^4s^2 - bc^2s^2 + 2c^2bs - b}}{c^2s^2 - 2c^2s + 1}. \tag{26}$$

Summing up equations (18)–(26), we obtain the following result.

Theorem 6. *Let $\gamma_1 = \gamma(s)$ be a unit speed, regular C^n-continuous curve, and $\gamma_2 = \gamma'(s)$ is the other base curve. Then, the blending curve β has the parametrization of the form $\beta(s) = f(s)\gamma(s) + (1-f(s))\gamma'(s)$, $s \in [0,1]$. Furthermore, if the base curve γ is constant ratio of the form $\rho = \|\gamma(s)\| = cs$ and blending β is spherical (i.e., $\|grad\tilde{\rho}\| = 0$), then the blending function*

$$f(s) = \frac{\pm\left(c^2s-1\right) + \sqrt{1 - 2c^2s + c^4s^2 - bc^2s^2 + 2c^2bs - b}}{c^2s^2 - 2c^2s + 1}. \tag{27}$$

Remark 2. In Theorem 6, we have shown that if the blending curve is formed by a unit velocity curve and its tangent expansion is spherical, then the blending function must have the parametrization of the form (27). Note that in the proof of the theorem, the base curve is assumed to be constant ratio.

3.2. Examples of Blended Models of Two Parametric Curves. In this subsection, two arbitrarily given curves $\gamma_1(t)$ and $\gamma_2(t)$ are blended to a third curve $\beta(t)$ so that $\beta(t)$ joins γ_1 and γ_2 in given points p_1 and p_2, C^m-continuously. In order to meet this objective one can use polynomial functions for the blending process [28].

Our construction shall be a linear blending process between the curves $\gamma_1(t)$ and $\gamma_2(t)$ (see Figure 1). In order to keep the construction simple, we will use polynomial functions. The advantage of using polynomial blending functions is that the blending curve $\beta(t)$ is polynomial if the curves $\gamma_1(t)$ and $\gamma_2(t)$ are polynomial. A disadvantage of the uniquely defined blending functions is the lack of design parameters.

Usually, blending formulas are arranged so that the blend curve matches each blender curve at one point. The intent is to provide a transition from one blender curve to the other blender curve. The number of derivatives of the curves that match at their endpoints is generally quite limited, so when talking about smoothness, it is the amount of matching at the endpoints that is being considered [29].

However, in order to talk about blending curves, some blending curves are considered using some polynomial blending functions and are given as follows:

i. G^0-blending: for the value $f(t) = 1-t$, C^1 is blending function, and one can get $f(0) = 1$, $f(1) = 0$. In this case, the blend curve has the parametrization

$$\beta(t) = (1-t)\gamma_1(t) + t\gamma_2(t), \quad t \in [0,1], \tag{28}$$

passing through the points $\beta(0) = \gamma_1(0) = p_1$, $\beta(1) = \gamma_2(1) = p_2$.

ii. G^1-blending: for the value $f(t) = 1 - 3t^2 + 2t^3$, C^2 is blending function and one can get

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$$\begin{aligned} f(0) &= 1, \\ f(1) &= 0, \\ f'(0) &= f'(1) = 0. \end{aligned} \tag{29}$$

In this case, the blend curve has the parametrization

$$\beta(t) = \left(1 - 3t^2 + 2t^3\right)\gamma_1(t) + \left(3t^2 - 2t^3\right)\gamma_2(t), \quad t \in [0,1], \tag{30}$$

where

$$\begin{aligned} \beta(0) &= \gamma_1(0) = p_1, \\ \beta(1) &= \gamma_2(1) = p_2 \\ \beta'(0) &= \gamma_1{}'(0), \\ \beta'(1) &= \gamma_2{}'(1). \end{aligned} \tag{31}$$

iii. G^2-blending: for the value $f(t) = 1 - (6t^5 - 15t^4 + 10t^3)$, C^2 is blending function and one can get

$$\begin{aligned} f(0) &= 1, \\ f(1) &= 0, \\ f'(0) &= f'(1) = 0, \\ f''(0) &= f''(1) = 0. \end{aligned} \tag{32}$$

In this case, the blend curve has the parametrization

$$\beta(t) = \left(1 - 6t^5 + 15t^4 - 10t^3\right)\gamma_1(t) + \left(6t^5 - 15t^4 + 10t^3\right)\gamma_2(t), \tag{33}$$

where

$$\begin{aligned} \beta(0) &= \gamma_1(0) = p_1, \\ \beta(1) &= \gamma_2(1) = p_2, \\ \beta'(0) &= \gamma_1{}'(0), \\ \beta'(1) &= \gamma_2{}'(1), \\ \beta''(0) &= \gamma_1{}''(0), \\ \beta''(1) &= \gamma_2{}''(1). \end{aligned} \tag{34}$$

Example 1. Let γ_1 and γ_2 be a parabola and spiral given with the parametrizations

$$\begin{aligned} \gamma_1(t) &= \left(t, t^2\right), \\ \gamma_2(t) &= (t\cos t, t\sin t), \end{aligned} \tag{35}$$

respectively.

i. The G^0-blending of γ_1 with γ_2 is given by

$$\beta(t) = \left(t(1-t) + t^2\cos t, t^2(1-t) + t^2\sin t\right), \tag{36}$$

(see Figure 2(a)).

ii. The G^1-blending of γ_1 with γ_2 is given by

$$\beta(t) = \left(\left(1 - 3t^2 + 2t^3\right)t + \left(3t^2 - 2t^3\right)t\cos t, \left(1 - 3t^2 + 2t^3\right)t^2 + \left(3t^2 - 2t^3\right)t\sin t\right), \tag{37}$$

(see Figure 2(b)).

iii. The G^2-blending of γ_1 with γ_2 is given by

$$\beta(t) = \begin{pmatrix} \left(1 - 6t^5 + 15t^4 - 10t^3\right)t + \left(6t^5 - 15t^4 + 10t^3\right)t\cos t, \\ \left(1 - 6t^5 + 15t^4 - 10t^3\right)t^2 + \left(6t^5 - 15t^4 + 10t^3\right)t\sin t, \end{pmatrix} \tag{38}$$

(see Figure 2(c)).

Example 2. Consider the base curve γ_1 and γ_2 given with the parametrizations

$$\begin{aligned} \gamma_1(t) &= \left(\cos\left(t^2 + 1\right), \sin\left(t^2 + 1\right)\right), \\ \gamma_2(t) &= \gamma_1{}'(t) = 2t\left(-\sin\left(t^2 + 1\right), \cos\left(t^2 + 1\right)\right). \end{aligned} \tag{39}$$

We consider the following cases:

i. The G^0-blending of γ_1 with γ_2 is given by

$$\beta(t) = \left(t\cos\left(t^2 + 1\right) - 2t(1-t)\sin\left(t^2 + 1\right), t\sin\left(t^2 + 1\right) + 2t(1-t)\cos\left(t^2 + 1\right)\right), \tag{40}$$

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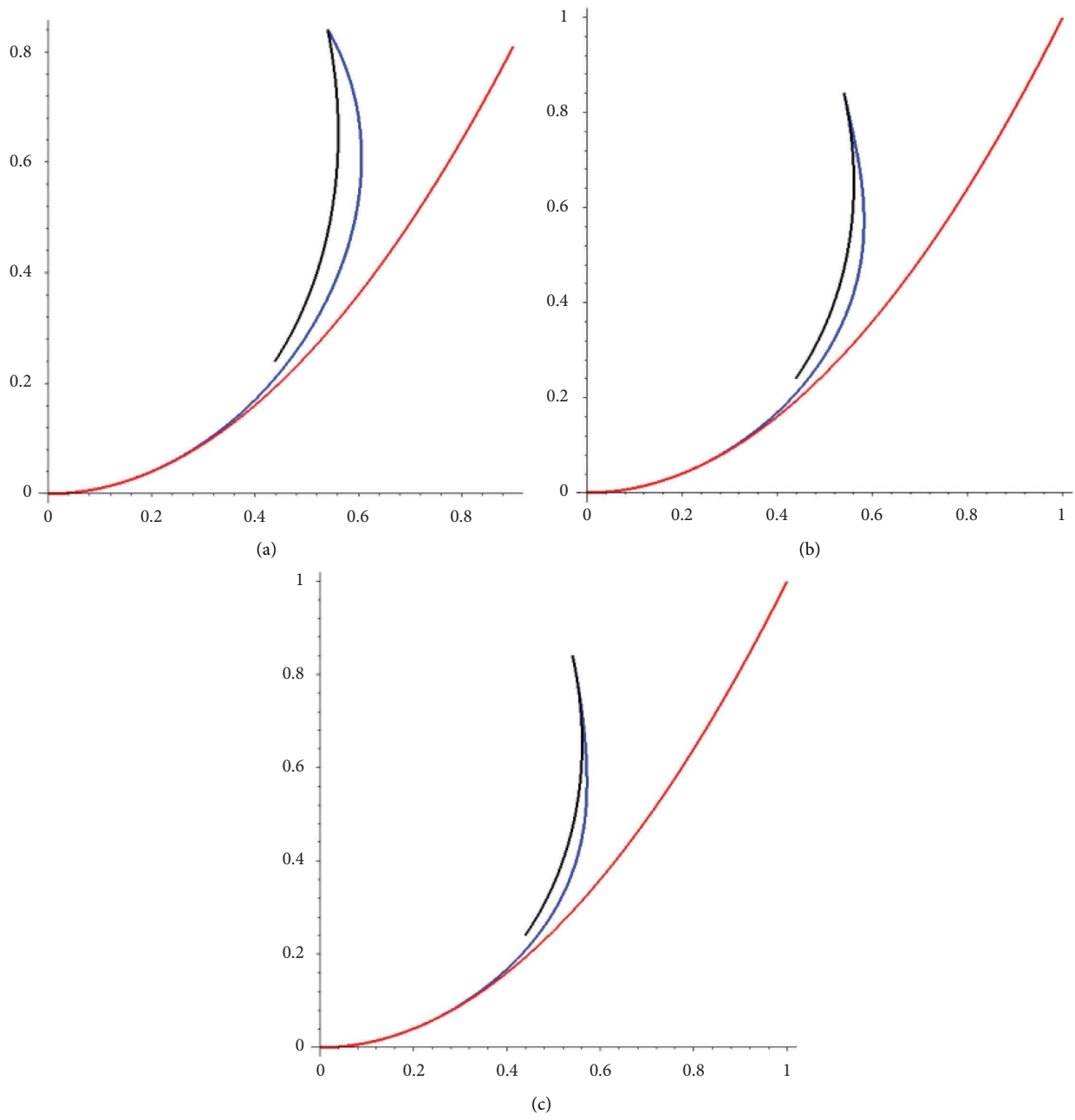

FIGURE 2: Blending (blue) of parabola (red) with spiral (black). (a) G^0–blending. (b) G^1–blending. (c) G^2–blending.

(see Figure 3(a)).

ii. The G^1-blending of γ_1 with γ_2 is given by

$$\beta(t) = \begin{pmatrix} \left(1 - 3t^2 + 2t^3\right)\cos\left(t^2 + 1\right) - 2t\left(3t^2 - 2t^3\right)\sin\left(t^2 + 1\right), \\ \left(1 - 3t^2 + 2t^3\right)\sin\left(t^2 + 1\right) + 2t\left(3t^2 - 2t^3\right)\cos\left(t^2 + 1\right), \end{pmatrix} \tag{41}$$

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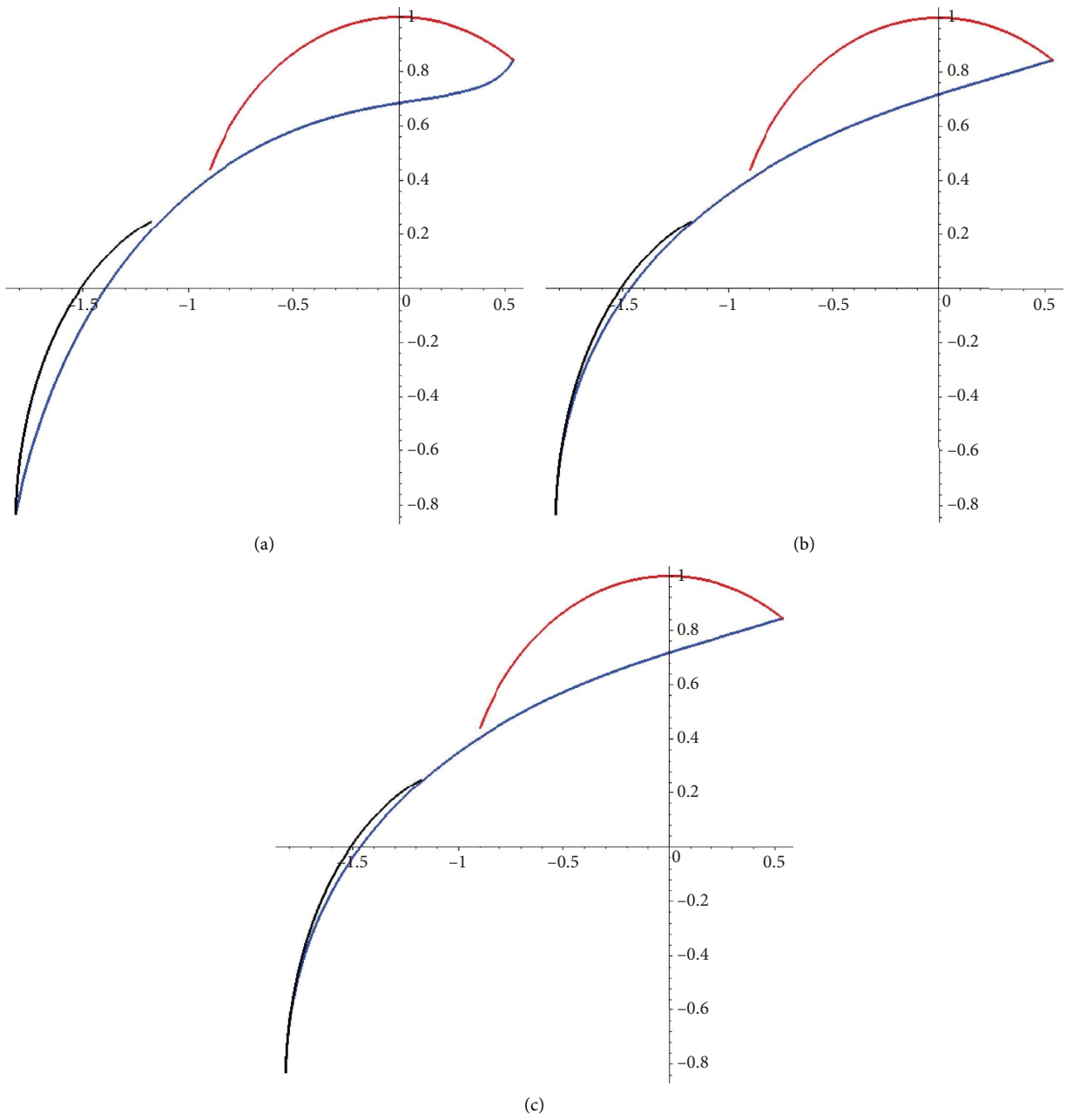

FIGURE 3: Blending (blue) of circular arc (red) with its tangent curve (black). (a) G^0–blending. (b) G^1–blending. (c) G^2–blending.

(see Figure 3(b)).

iii. The G^2-blending of γ_1 with γ_2 is given by (see Figure 3(c)):

$$\beta(t) = \begin{pmatrix} \left(1 - 6t^5 + 15t^4 - 10t^3\right)\cos\left(t^2 + 1\right) - 2t\left(6t^5 - 15t^4 + 10t^3\right)\sin\left(t^2 + 1\right), \\ \left(1 - 6t^5 + 15t^4 - 10t^3\right)\sin\left(t^2 + 1\right) + 2t\left(6t^5 - 15t^4 + 10t^3\right)\cos\left(t^2 + 1\right) \end{pmatrix}. \tag{42}$$

3.3. Constant-Ratio Blended Models of Two Parametric Surfaces. A blending surface is a surface that smoothly connects two given surfaces along two arbitrary curves, one on each surface. This is particularly useful in the modeling operations of filleting a sharp edge between joining surfaces or connecting disjoint surfaces.

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Blending method can be used to design many interesting curves, surfaces, and objects. Some types of blending methods are linear, quadratic, and cubic blending. These methods have similar concept to Bézier because the blending method only touch the first and last curve or surface without touching the intermediate curves or surfaces. Surface blending methods can produce some blended surfaces or object [30].

In this subsection, we derive a new constant-ratio surface for representing surfaces, which are blends of parametric surfaces, i.e., surface of revolution.

Let $X = X(s,t)$ be a regular C^n-continuous surface patch of a surface of revolution

$$X(s,t) = (g(t)\cos s, g(t)\sin s, h(t)), \tag{43}$$

and $Y = X_t(s,t)$ is the tangent indicatrix of X, which is the other base surface. Then, the blending surface of X with Y has the parametrization

$$B(s,t) = f(t)X(s,t) + (1 - f(t))X_t(s,t), \quad t \in [0,1]. \tag{44}$$

The distance function of the blending surface B becomes

$$\tilde{\rho} = \|B(s,t)\| = \sqrt{f^2(t)\langle X(s,t), X(s,t)\rangle + (1-f(t))^2 + 2f(t)(1-f(t))\langle X(s,t), X_t(s,t)\rangle}. \tag{45}$$

By the use of

$$\rho = \|X(s,t)\| = \sqrt{g^2(t) + h^2(t)}, \tag{46}$$

with

$$\begin{aligned} \langle X(s,t), X_t(s,t)\rangle &= g(t)g'(t) + h(t)h'(t) = \rho\rho', \\ \langle X(s,t), X_s(s,t)\rangle &= 0, \end{aligned} \tag{47}$$

one can get

$$\tilde{\rho} = \sqrt{f^2(t)\rho^2 + (1-f(t))^2 + 2f(t)(1-f(t))\rho\rho'}. \tag{48}$$

The gradient of the distance function ρ is defined by

$$\text{grad}\rho = \sum_{j=1}^{2} \frac{\langle X(s,t), e_j\rangle}{\|X(s,t)\|} e_j, \tag{49}$$

where e_1, e_2 are orthonormal tangent vectors of the first base surface.

$$\begin{aligned} e_1 &= \frac{X_s(s,t)}{\|X_s(s,t)\|}, \\ e_2 &= \frac{X_t(s,t)}{\|X_t(s,t)\|}. \end{aligned} \tag{50}$$

Thus, we get

$$\begin{aligned} \|\text{grad}\rho\| &= \frac{\langle X(s,t), e_1\rangle + \langle X(s,t), e_2\rangle}{\rho} \\ &= \frac{g(t)g'(t) + h(t)h'(t)}{\sqrt{g^2(t) + h^2(t)}}. \end{aligned} \tag{51}$$

Assume that the base surface is constant ratio of the form $\rho = \|X(s,t)\| = ct$. Then,

$$\tilde{\rho} = \sqrt{f^2(t)c^2t^2 + (1-f(t))^2 + 2c^2f(t)(1-f(t))t} \tag{52}$$

holds. Furthermore, if the blending surface is spherical, i.e., $\|\text{grad}\tilde{\rho}\| = 0$ holds identically, then

$$0 = \|\text{grad}\tilde{\rho}\| = \frac{\langle B(s,t), \tilde{e}_1\rangle + \langle B(s,t), \tilde{e}_2\rangle}{\tilde{\rho}}, \tag{53}$$

where $\tilde{e}_1$ and $\tilde{e}_2$ are orthonormal tangent vectors of the blending surface.

$$\begin{aligned} \tilde{e}_1 &= \frac{B_s(s,t)}{\|B_s(s,t)\|}, \\ \tilde{e}_2 &= \frac{B_t(s,t)}{\|B_t(s,t)\|}. \end{aligned} \tag{54}$$

Since $\langle B(s,t), B_s(s,t)\rangle = 0$, then from equation (53), we get $\langle B(s,t), B_t(s,t)\rangle = 0$. Consequently, differentiating equation (44) with respect to t and using the equation above, we obtain the following differential equation:

$$(1-f(t))\left(c^2f(t) + \left(c^2t - 1\right)f'(t)\right) + f(t)c^2t\left(f'(t)(t-1) + 1\right) = 0. \tag{55}$$

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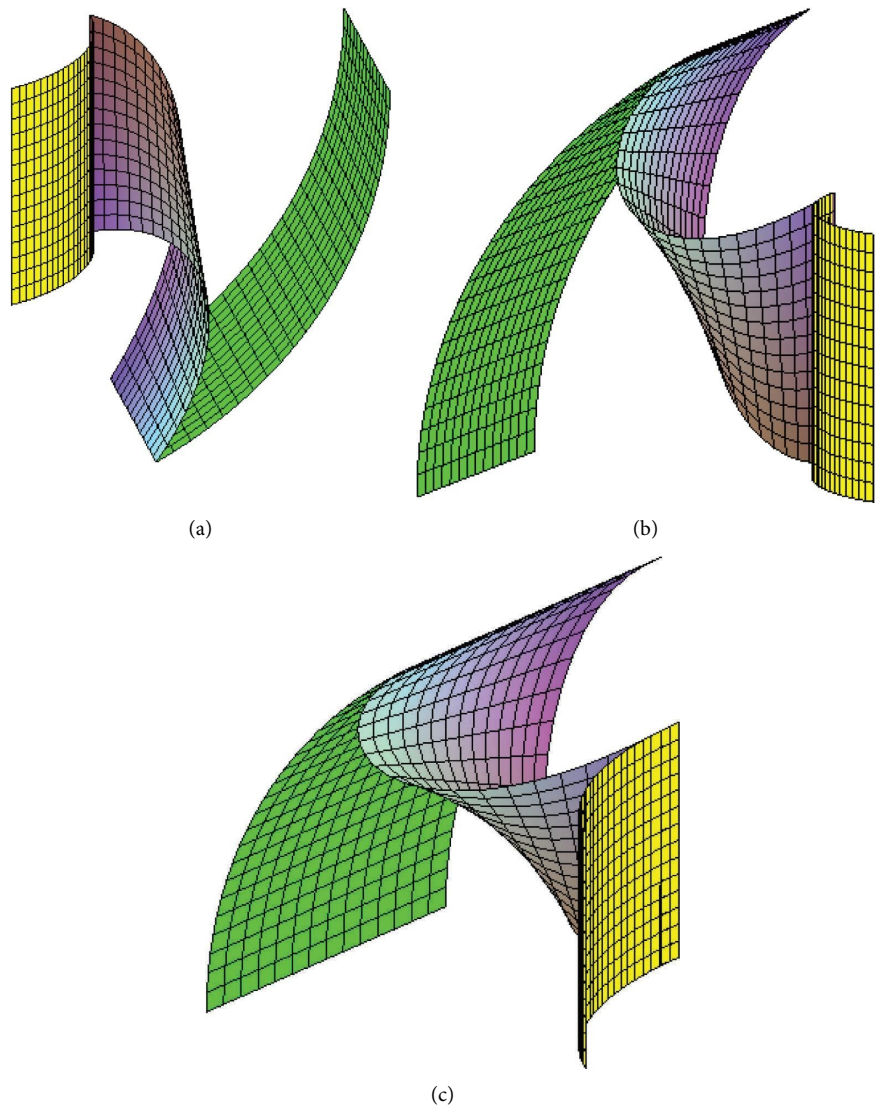

(a) (b) (c)

FIGURE 4: Blending surface of two cylinders with given blend functions. (a) $f(t) = 1 - t$. (b) $f(t) = 1 - 3t^2 + 2t^3$. (c) $f(t) = 1 - 6t^5 + 15t^4 - 10t^3$.

Solving this differential equation with Maple, one can get

$$f(t) = \frac{1 - c^2 t + \sqrt{1 - 2c^2 t + c^4 t^2 - bc^2 t^2 + 2c^2 bt - b}}{c^2 t^2 - 2c^2 t + 1}, \tag{56}$$

or

$$f(t) = \frac{-1 + c^2 t + \sqrt{1 - 2c^2 t + c^4 t^2 - bc^2 t^2 + 2c^2 bt - b}}{c^2 t^2 - 2c^2 t + 1}. \tag{57}$$

Hence, summing up equations (43)–(57), we obtain the following result.

Theorem 7. *Let $X = X(s,t)$ be a surface patch of a surface of revolution and $Y = X_t(s,t)$ is the other base surface. Then, the blending surface has the parametrization of the form*

$$B(s,t) = f(t)X(s,t) + (1 - f(t))X_t(s,t), \quad t \in [0,1]. \tag{58}$$

Furthermore, if the base surface X is constant ratio of the form $\rho = \|X(s,t)\| = ct$ and blending surface B is spherical (i.e., $\|\text{grad}\tilde{\rho}\| = 0$), then the blending function becomes

$$f(t) = \frac{\pm\left(c^2 t - 1\right) + \sqrt{1 - 2c^2 t + c^4 t^2 - bc^2 t^2 + 2c^2 bt - b}}{c^2 t^2 - 2c^2 t + 1}. \tag{59}$$

Remark 3. In Theorem 7, we have shown that if the blending surface formed by a surface of revolution and its tangent expansion is spherical, then the blending function must have the parametrization of the form (59). Note that in the proof

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of the theorem, the base surface is assumed to be constant ratio.

Example 3. Let X_1 and X_2 represent two cylindrical surfaces given with the parametrizations

$$\begin{aligned} X_1(s,t) &= (\cos t,\ \sin t,\ s),\\ X_2(s,t) &= (2\cos t,\ s,\ 2\sin t), \end{aligned} \tag{60}$$

respectively.

i. The G^0-blending surface of X_1 with X_2 is given with the parametrization

$$X(s,t) = ((1-t)\cos t + 2t\cos t,\ (1-t)\sin t + st,\ (1-t)s + 2t\sin t), \tag{61}$$

where the blending function is the polynomial $f(t) = 1 - t$ (Figure 4(a)).

ii. The G^1-blending surface of X_1 with X_2 is given with the parametrization

$$X(s,t) = \begin{pmatrix} \left(1-3t^2+2t^3\right)\cos t + 2\left(3t^2-2t^3\right)\cos t,\\ \left(1-3t^2+2t^3\right)\sin t + \left(3t^2-2t^3\right)s,\\ \left(1-3t^2+2t^3\right)s + 2\left(3t^2-2t^3\right)\sin t, \end{pmatrix}, \tag{62}$$

where the blending function is the polynomial $f(t) = 1 - 3t^2 + 2t^3$ (Figure 4(b)).

iii. The G^2-blending surface of X_1 with X_2 is given with the parametrization

$$X(s,t) = \begin{pmatrix} \left(1-6t^5+15t^4-10t^3\right)\cos t + 2\left(6t^5-15t^4+10t^3\right)\cos t,\\ \left(1-6t^5+15t^4-10t^3\right)\sin t + \left(6t^5-15t^4+10t^3\right)s,\\ \left(1-6t^5+15t^4-10t^3\right)s + 2\left(6t^5-15t^4+10t^3\right)\sin t, \end{pmatrix}, \tag{63}$$

where the blending function is polynomial $f(t) = 1 - 6t^5 + 15t^4 - 10t^3$ (Figure 4(c)).

4. Conclusion

In this section, we can give the evaluation in several parts. First, the methodology summary, then the applications, the comments that are the source of future studies, and finally discussion of choosing this method are mentioned.

4.1. Methodology Summary. By the use of Hartmann's simple blending method, we produced some parametric blending curves and surfaces that have G^n-contact with the base curves and surfaces. This method is used to create parametric blending curves and surfaces that maintain contact with the base curves or surfaces. It is practical for design as it allows for smooth transitions between different geometries. The design parameters control how these curves or surfaces blend together, influencing the shape and continuity of the resulting geometry. We also constructed the blending curve (or surface) by choosing the circular curve (or rotational surface) and the tangents indicator of this curve (or surface) as the basis curves.

4.2. Applications. By using circular curves or rotational surfaces as the basis for blending, we can achieve specific geometric properties in the resulting blended curve or surface. For instance, ensuring a constant ratio in the blending function can lead to uniform curvature along the blended surface. The blending of constant ratio curves and surfaces is an important concept in computer-aided geometric design (CAGD), computer graphics, and engineering applications such as CAD/CAM, animation, and manufacturing. Blending constant ratio curves and surfaces plays a vital role in geometric modeling by ensuring smoothness, efficiency, and precision in various design and engineering applications.

4.3. Future Work. Many solid modeling systems have been developed and are widely used in practical engineering fields today because solid models are much more preferred than surface models and wire frame models for various applications. However, the solids that these systems can produce are generally polyhedra with simple surfaces.

If we blend two solid parts, it is possible to get a complex solid object. Thus, rotational surfaces and blending surfaces obtained with the help of blending curves can serve as examples of solid modeling. Incorporating free-form surface

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modeling capabilities into solid modeling systems is crucial for handling complex geometries that cannot be represented adequately by simple polyhedra. As a result, blending curves and surfaces play a significant role in geometric design and solid modeling by enabling smooth transitions and complex shapes. These techniques are essential for creating realistic and functional solid objects in engineering and design fields.

4.4. Discussion. In this study, blending is done for curves and surfaces with constant ratio. The reason we choose this is to use these curve and surface models that give important geometric results in differential geometry by taking the most basic blending functions. However, it is likely to cause some complexity and difficulties in terms of implementation or flexibility.

While constant-ratio functions offer elegant theoretical properties and are valuable in modeling smooth, self-similar curves, their applications (especially in engineering, design, and computational geometry) come with some practical and theoretical limitations. Some of these are limited expression for complex geometries, increased computational cost in higher dimensions, lack of flexibility in design, and simplification in physical modeling. Therefore, for complex modeling tasks, more flexible approaches such as spline curves, subdivision surfaces, or curvature-driven optimization methods may be required to overcome the rigidity and computational difficulties inherent in constant-rate formulations.

Data Availability Statement

Data sharing is not applicable as no new data were generated.

Conflicts of Interest

The author declares no conflicts of interest.

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References

[1] E. Hartmann, "Implicit G^n-Blending of Vertices," *Computer Aided Geometric Design* 18, no. 3 (2001): 267–285, https://doi.org/10.1016/s0167-8396(01)00030-9.

[2] E. Hartmann, "Parametric G^n-Blending of Curves and Surfaces," *The Visual Computer* 17 (2001): 1–13, https://doi.org/10.1007/pl00013398.

[3] P. Kiciak, "Bicubic B-Spline Blending Patches With Optimized Shape," *Computer-Aided Design* 43, no. 2 (2011): 133–144, https://doi.org/10.1016/j.cad.2010.10.003.

[4] Q. Song and J. Wang, "Generating G^n Parametric Blending Surfaces Based on Partial Reparameterization of Base Surfaces," *Computer-Aided Design* 39, no. 11 (2007): 953–963, https://doi.org/10.1016/j.cad.2007.06.009.

[5] R. E. Barnhill, G. E. Farin, and Q. Chen, "Constant-Radius Blending of Parametric Surfaces," in *Computing Supplementum*, ed. G. Farin, H. Noltemeier, H. Hagen, and W. Knödel, 8 (Springer, 1993), 1–20, https://doi.org/10.1007/978-3-7091-6916-2_1.

[6] B. Belkhatir and A. Zidna, "Construction of Flexible Blending Parametric Surfaces via Curves," *Mathematics and Computers in Simulation* 79, no. 12 (2009): 3599–3608, https://doi.org/10.1016/j.matcom.2009.04.015.

[7] I. C. Braid, "Non-Local Blending of Boundary Models," *Computer-Aided Design* 29, no. 2 (1997): 89–100, https://doi.org/10.1016/s0010-4485(96)00038-3.

[8] J. H. Chuang and W. C. Hwang, "Variable-Radius Blending by Constrained Spine Generation," *The Visual Computer* 13, no. 7 (1997): 316–329, https://doi.org/10.1007/s003710050106.

[9] R. A. Farouki and R. Sverrisson, "Approximation of Rolling-Ball Blends for Free-Form Parametric Surfaces," *Computer-Aided Design* 28, no. 11 (1996): 871–878, https://doi.org/10.1016/0010-4485(96)00008-5.

[10] T. Hermann, G. Lukacs, and T. Varady, in *Techniques for Variable Radius Rolling Ball Blends* (Vanderbilt University, 1995).

[11] G. Lukacs, "Differential Geometry of G^1 Variable Radius Rolling Ball Blend Surfaces," *Computer Aided Geometric Design* 15, no. 6 (1998): 585–613, https://doi.org/10.1016/s0167-8396(98)00006-5.

[12] T. Varady and A. Rockwood, "Geometric Construction for Setback Vertex Blending," *Computer-Aided Design* 26 (1997): 341–365.

[13] J. Wallner and H. Pottmann, "Rational Blending Surfaces Between Quadrics," *Computer Aided Geometric Design* 14, no. 5 (1997): 407–419, https://doi.org/10.1016/s0167-8396(96)00037-4.

[14] X. You, F. Tian, W. Tang, J. Chang, and J. Zhang, "Analytical C^2 Continuous Surface Blrnding," *Mathematics* 12, no. 19 (2024): 3096, https://doi.org/10.3390/math12193096.

[15] J. Cheng, "Blending Quadric Surfaces via Base Curve Method," *Computer Mathematics* 77–86 (2003).

[16] D. C. Douglas and M. Dimitri, "Blended Deformable Models," in *Proceedings of IEEE Conference on Computer Vision and Pattern Recognition* (Seattle, WA, USA, 1994), 566–572.

[17] J. Vida, R. R. Martin, and T. Varady, "A Survey of Blending Methods That Use Parametric Surfaces," *Computer-Aided Design* 26 (1994): 341–365.

[18] H. Chiyokura and F. Kimura, "Design of Solids With Free-Form Surfaces," *ACM SIGGRAPH Computer Graphics* 17, no. 3 (1983): 289–298, https://doi.org/10.1145/964967.801160.

[19] A. G. Requicha, "Representations for Rigid Solids: Theory, Methods, and Systems," *ACM Computing Surveys* 12, no. 4 (1980): 437–464, https://doi.org/10.1145/356827.356833.

[20] M. A. Sanglikar, P. Koparkar, and V. N. Joshi, "Modelling Rolling Ball Blends for Computer Aided Geometric Design," *Computer Aided Geometric Design* 7, no. 5 (1990): 399–414, https://doi.org/10.1016/0167-8396(90)90003-a.

[21] B. Y. Chen, "Constant Ratio Hypersurfaces," *Soochow Journal of Mathematics* 28 (2001): 353–362.

[22] S. Gürpınar, K. Arslan, and G. Öztürk, "A Characterization of Constant Ratio Curves in Euclidean Space," *Acta Universtatis Apulensis* 44 (2015): 39–51.

[23] B. Y. Chen, "More on Convolution of Riemannian Manifolds," *Beitrage zur Algebra und Geometrie* 43 (2002): 9–24.

[24] P. Koparkar, "Parametric Blending Using Fanout Surfaces," in *Proceedings of the First ACM Symposium on Solid Modeling Foundations and CAD/CAM Applications (SMA'91)* (New York, NY, USA: Association for Computing Machinery, 1991), 317–327.

[25] X. Ye, "Gc^n Continuity Condition for Adjacent Surfaces Patches," *Computer Aided Geometric Design* 13 (1996): 524–548.

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[26] M. L. Mazure, "Geometric Contact for Curves and Surfaces," *Computer Aided Geometric Design* 11, no. 2 (1994): 177–195, https://doi.org/10.1016/0167-8396(94)90031-0.
[27] K. C. Hui, "Shape Blending of Curves and Surfaces With Geometric Continuity," *Computer-Aided Design* 31, no. 13 (1999): 819–828, https://doi.org/10.1016/s0010-4485(99)00059-7.
[28] A. Wiltsche, "Blending Curves," *Journal for Geometry and Graphics* 9, no. 1 (2005): 67–75.
[29] D. S. Meek and D. J. Walton, "Blending Two Parametric Curves," *Computer-Aided Design* 41, no. 6 (2009): 423–431, https://doi.org/10.1016/j.cad.2009.02.006.
[30] S. J. Ismail and J. M. Ali, "Surface Design by Blending Rational Bezier Curves and Surfaces," in *AIP Conference Proceedings*, 1605 (2014), 274–279, https://doi.org/10.1063/1.4887601.

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