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SOME GRAPH PARAMETERS OF POWER SET GRAPHS

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Abstract

In this study, we examine some graph parameters such as the edge number, chromatic number, girth, domination number and clique number of power set graphs.

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1. Introduction

Graph theory is very important as it is being actively applied in computer science, biochemistry, nanotechnology and electrical engineering [8, 13, 19 20]. Research on graphs associated with a finite algebraic structure has attracted a lot of attention. The examples of a graph obtained from an algebraic structure are identity graphs, [17], Cayley graphs, [16], annihilator graphs, [3] and [7], zero divisor graphs, [4-6] and [9], semigroup graphs, [1, 2, 11, 12] and [21], etc. For graph theoretical notations and terminologies, see [10, 14] and [15].

The power set graph was defined in [18]. The elements of the vertex set of the power set graph are the elements of the power set of A, that is the subsets of A. Since $\varnothing$ and A are adjacent to all other vertices in the power set graph, we can neglect these two vertices. So, we define the power set graph as follows: Let $P(A)$ be the power set of a set A. The edge and vertex sets of the graph $\Gamma(P(A))$ are denoted by $E(\Gamma(P(A)))$ and $V(\Gamma(P(A)))$, respectively, where $V(\Gamma(P(A))) = \{X : X \in P(A)$ and $X \neq \varnothing, A\}$. Any two vertices X, Y of $\Gamma(P(A))$ are adjacent if and only if $X \subset Y$ or $Y \subset X$.

In this study, we examine some graph parameters such as the edge number, chromatic number, girth, domination number and clique number of power set graphs.

2. Main Results

Let $\Gamma(P(A))$ be the power set graph corresponding to a set A. Then the degree of a vertex X of $\Gamma(P(A))$ is defined from the following theorem.

Theorem 2.1. *Let X be a non-empty proper subset of A. Then degree of X in $\Gamma(P(A))$ is*

$$\deg(X) = 2^{|X|} + 2^{|A|-|X|} - 4. \tag{1}$$

Proof. Let us assume that $X = \{1, 2, 3, ..., k\}$ and $A = \{1, 2, 3, ..., n\}$ for $1 \le k \le n-1$. First of all, the vertex X is adjacent to the vertices corresponding to its proper subsets. We can easily determine the number of such neighbors as $2^k - 2$. Secondly, the vertex X is adjacent to the vertices corresponding to the supersets Y so that $X \subset Y$, $Y \neq \varnothing$ and $Y \subset A$. The number of such neighbors is $2^{n-k} - 2$. So we get

$$\deg(X) = 2^k - 2 + 2^{n-k} - 2 = 2^k + 2^{n-k} - 4$$

as required. □

Theorem 2.2. *The maximum degree and the minimum degree of* $\Gamma(P(A))$ *are as follows*:

$$\Delta(\Gamma(P(A))) = 2^{|A|-1} - 2, \quad \Delta(\Gamma(P(A))) = 2^{\left\lfloor \frac{|A|+1}{2} \right\rfloor} + 2^{\left\lfloor \frac{|A|}{2} \right\rfloor} - 4.$$

Proof. By Theorem 2.1, we know that the degree of a vertex X in graph $\Gamma(P(A))$ is $\deg(X) = 2^{|X|} + 2^{|A|-|X|} - 4$. Now consider the function

$$f(x) = 2^x + 2^{|A|-x} - 4.$$

It can easily be shown that $f(x)$ is decreasing for $x \le \left\lfloor \frac{|A|}{2} \right\rfloor$ and is increasing for $x \ge \left\lfloor \frac{|A|}{2} \right\rfloor$. So we get

$$f(x) \ge f\left(\left\lfloor \frac{|A|}{2} \right\rfloor\right) = 2^{|X|} + 2^{|A|-|X|} - 4.$$

Considering this, we get $\delta(\Gamma(P(A))) = 2^{\left\lfloor \frac{|A|+1}{2} \right\rfloor} + 2^{\left\lfloor \frac{|A|}{2} \right\rfloor} - 4$. By similar argument, we can easily see that

$$f(x) \le f(1) = f(|A| - 1) = 2^1 + 2^{|A|-1} - 4.$$

Considering this, we get $\Delta(\Gamma(P(A))) = 2^{|A|-1} - 2$. □

Theorem 2.3. *The number of edges of* $\Gamma(P(A))$ *is* $3^{|A|} - 3.2^{|A|} + 3$.

Proof. Let $|A| = n$. We know that the sum of vertex degrees in a graph is twice the number of edges by the handshaking lemma. Then by Theorem 2.1, we have

$$\sum_{X \in V(\Gamma(P(A)))} \deg(X)$$

$$= \sum_{\substack{X \in V(\Gamma(P(A))) \\ |X|=1}} \deg(X) + \sum_{\substack{X \in V(\Gamma(P(A))) \\ |X|=2}} \deg(X) + \cdots + \sum_{\substack{X \in V(\Gamma(P(A))) \\ |X|=n-1}} \deg(X)$$

$$= \binom{n}{1}[2^{n-1} + 2 - 4] + \binom{n}{2}[2^{n-2} + 2^2 - 4]$$

$$+ \cdots + \binom{n}{n-1}[2^{n-(n-1)} + 2^{n-1} - 4]$$

$$= \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} + \cdots + \binom{n}{n-1}2^1$$

$$+ \binom{n}{1}2^1 + \binom{n}{2}2^2 + \cdots + \binom{n}{n-1}2^{n-1} - 4\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1}$$

$$= 2\left[(2+1)^n - \binom{n}{0}2^n - \binom{n}{n}1^n\right] - 4\left[2^n - \binom{n}{0} - \binom{n}{n}\right]$$

$$= 2[3^n - 3.2^n + 3].$$

Then we get $|E(\Gamma(P(A)))| = 3^n - 3.2^n + 3$, as desired. □

Theorem 2.4. *The graph* $\Gamma(P(A))$ *is a* 3*-self centered graph*.

Proof. Let *A* be a non-empty set. Let *X* be a non-empty proper subset of the set *A*. We have $X^c = A - X$. Thus, we can easily see that the farthest vertex to the vertex *X* is the vertex X^c. Therefore, we get $d(X, X^c) = 3$

since $X \sim X \cup \{i\} \sim \{i\} \sim X^c$ for $i \notin X$. In other words, $ecc(X) = 3$. So we get $ecc(X) = 3$ for all subsets X of A, as desired. □

The girth of a graph G with a cycle is the length of its shortest cycle. An acyclic graph has infinite girth. Next, we calculate the girth of the power set graphs:

Theorem 2.5. $girth\,(\Gamma(P(A))) = 3$.

Proof. Let $A = \{i_1, i_2, ..., i_n\}$ be a non-empty set. The vertex $\{i_1\}$ in $\Gamma(P(A))$ is adjacent to the vertex $\{i_1, i_2, i_3\}$ by definition. Also, the vertex $\{i_1, i_2, i_3\}$ is adjacent to the vertex $\{i_1, i_3\}$. Moreover, the vertex $\{i_1, i_3\}$ is adjacent to the vertex $\{i_1\}$. That is $\{i_1\} \sim \{i_1, i_2, i_3\} \sim \{i_1, i_3\} \sim \{i_1\}$. Thus girth of $\Gamma(P(A))$ is three. □

A set D of vertices of a graph G is called *dominating* if every vertex $u \in V(G)\backslash D$ is adjacent to a vertex of D. The domination number $\gamma(G)$ of a graph G is the size of a smallest dominating set of G. A dominating set D with $|D| = \gamma(G)$ is called a *minimum dominating set*. For instance, $\gamma(K_n) = 1$ and $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

We now determine the domination number of power set graphs:

Theorem 2.6. $\gamma(\Gamma(P(A))) = n$.

Proof. Let $A = \{i_1, i_2, ..., i_n\}$, and B be a non-empty proper subset of A. Then $\{i_k\} \sim B \cup \{i_k\}$ for $k = 1, 2, ..., n$. Since the vertex $\{i_k\}$ is not adjacent to the vertex $\{i_j\}$ for $j \in \{1, 2, ..., n\}\backslash\{k\}$ in $\Gamma(P(A))$, we get the dominating set as $D(\Gamma(P(A))) = \{i_1, i_2, ..., i_n\}$. Therefore, $\gamma(\Gamma(P(A)))$ $= n$. □

A coloring of a graph is an assignment of colors to its points so that no two adjacent points have the same color. The set of all points having the

same color is clearly independent and is called a *color class*, [15]. An *n-coloring* of a graph G is a coloring with n colors; it thereby partitions V into n color classes. The chromatic number $\chi(G)$ is defined as the minimum n for which G has an n-coloring.

We now obtain the chromatic number of $\Gamma(P(A))$.

Theorem 2.7.

$$\chi(\Gamma(P(A))) = n - 1.$$

Proof. Let $A = \{i_1, i_2, ..., i_n\}$. Let the color c_1 be used to color the vertex $\{i_1\}$. Since the vertex $\{i_1\}$ is not adjacent to the vertex $\{i_j\}$ in $\Gamma(P(A))$ for $j \in \{2, ..., n\}$, we can use the same color c_1 for the $\{i_j\}$ vertices. Next, let us take the vertex $\{i_1, i_2\}$. Since this vertex is adjacent to the vertex $\{i_1\}$, we cannot use the color c_1 for the vertex $\{i_1, i_2\}$. Therefore, we choose color c_2 for $\{i_1, i_2\}$. It is obvious that this vertex is not adjacent to the vertex $\{i_{n_1}, i_{n_2}\}$, where $\{i_{n_1}, i_{n_2}\} \subset A$. So we can use the same color c_2 for the $\{i_{n_1}, i_{n_2}\}$ vertices. If we continue this way, then we can use color c_{n-1} for the vertex $\{i_1, i_2, ..., i_{n-1}\}$. Hence, we obtain

$$\chi(\Gamma(P(A))) = n - 1. \qquad \square$$

A simple graph G in which each pair of distinct vertices are adjacent is a *complete graph*. The *clique number* of a graph G, denoted by $\omega(G)$, is the maximum number of vertices in a complete subgraph of G. The clique number of power set graphs is obtained below:

Theorem 2.8.

$$\omega(\Gamma(P(A))) = n - 1.$$

Proof. Let us take the set B as a subset of the power set $P(A)$, where

$$B = \{\{i_1\}, \{i_1, i_2\}, \{i_1, i_2, i_3\}, ..., \{i_1, i_2, ..., i_{n-1}\}\}.$$

We easily see that B is a maximal complete subgraph of $\Gamma(P(A))$. Then we get $\omega(\Gamma(P(A))) = n - 1$. So the proof is completed. □

Remark 2.9. By Theorems 2.7 and 2.8, we get

$$\omega(\Gamma(P(A))) = \chi(\Gamma(P(A))) = n - 1$$

which implies that the perfectness property for graphs $\Gamma(P(A))$ is preserved [14].

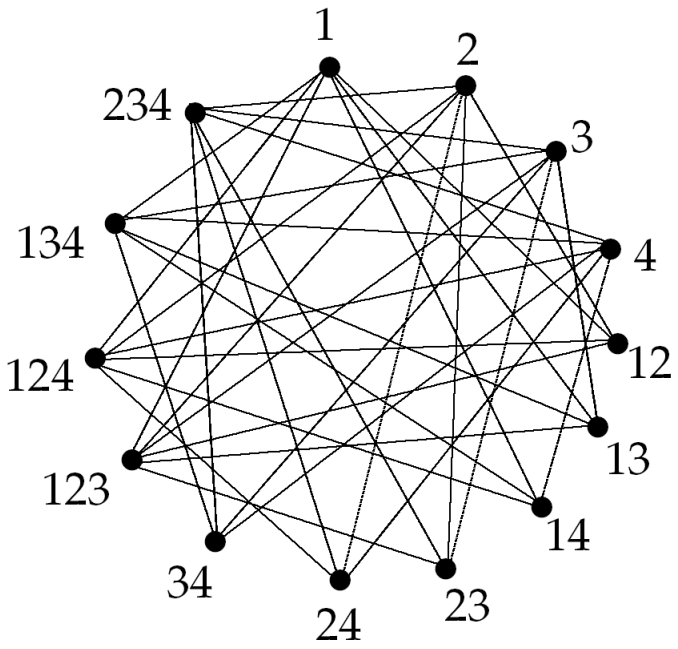


Figure 1. The graph $\Gamma(P(A))$.

Example 2.10. Let the graph $\Gamma(P(A))$ be as given in Figure 1, where $A = \{1, 2, 3, 4\}$. Then by the above results, we can state the following:

(1) $diam(\Gamma(P(A))) = 3$,

(2) $rad(\Gamma(P(A))) = 3$,

(3) $girth(\Gamma(P(A))) = 3$,

(4) $\Delta(\Gamma(P(A))) = 6$,

(5) $\delta(\Gamma(P(A))) = 4$,

(6) $\gamma(\Gamma(P(A))) = 4$,

(7) $\chi(\Gamma(P(A))) = 3$,

(8) $\omega(\Gamma(P(A))) = 3$.

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