Classification of Normal Subgroups of Hecke Group H6 in Terms of Parabolic Class Number

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Abstract. In [3], Greenberg showed that $n \leq 6t^3$ so that $\mu = nt \leq 6t^4$ for a normal subgroup N of level n and index μ having t parabolic classes in the modular group Γ. Accola, [1], improved these to $n \leq 6t^2$ always and $n \leq t^2$ if Γ/N is not abelian. Newman, [5], obtained another generalisation of these results. Hecke groups are generalisations of the modular group. We particularly deal with one of the most important cases, $q = 6$.

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INTRODUCTION

Hecke groups $H(\lambda_q)$ are the discrete subgroups of $PSL(2,\mathbf{R})$ generated by two elliptic elements

$$R(z) = \frac{-1}{z} \quad and \quad S(z) = -\frac{1}{z+\lambda_q}$$

where $\lambda_q = 2\cos\frac{\pi}{q}, q \geq 3, q \in \mathbf{N}$. As a Fuchsian group, they have the signature

$$(0; 2, q, \infty).$$

We let $T = RS$. For more details, see [2].

When $q = 3$, the well-known modular group denoted by Γ is obtained. The other Hecke groups can be considered as some kind of generalisation of Γ.

We define the level n of a normal subgroup N of $H(\lambda_q)$ as the least natural number so that $T^n = I$ in N. The parabolic class number t of N is defined to be the number of conjugacy classes of ∞. Basically, t is the number of infinities in the signature of N. For example, Hecke groups $H(\lambda_q)$ has $t = 1$.

Recall that the numbers μ, n and t are related to each others by the following formula $\mu = n \cdot t$, see [4].

For even values of q, when $t = 2$, we can always map onto $D_n \cong (2,2,n), n \in \mathbf{N}$, and this gives an infinite class of normal subgroups $W_n(\lambda_q) \cong (0; 2^{(n)}, \infty, \infty)$ having two parabolic classes. This shows that when q is even, $H(\lambda_q)$ does not necessarily have finitely many normal subgroups having a given number t of parabolic classes. But for some values of t it will still be possible to obtain the same result that we obtain for odd q, i.e. $H(\lambda_q)$ will have only finitely many normal subgroups of finite index having a fixed number t of parabolic classes for some values of t.

We now discuss our problem of finding the number of normal subgroups of the Hecke group $H(\lambda_6) = H(\sqrt{3})$ having a given number t of parabolic classes for small t.

THE CASE $q = 6$.

Here we discuss the cases $t = 1,\ 2,\ 3$, and 6.

i. $t = 1$: In this case it can be shown that there is no free cycloidal normal subgroup of $H(\sqrt{3})$, by the Riemann-Hurwitz formula. However there are some cycloidal normal subgroups having torsion corresponding to some cyclic quotients of $H(\sqrt{3})$. Actually there are only six of them with indices ≤ 6 (see [2]). Therefore

Theorem 1 *$H(\sqrt{3})$ has only finitely many normal subgroups, all with torsion, having one parabolic class.*

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ii. $t = 2$: In this case we have

Theorem 2 *$H(\sqrt{3})$ has infinitely many normal subgroups having two parabolic classes.*

Proof The proof follows because of the existence of the class of normal subgroups $W_n(\sqrt{3})$, similarly to the case $q = 4$.

iii. $t = 3$: Then $\mu = 3n$. Again as we are not interested in the cyclic quotients of $H(\sqrt{3})$ we map R to an element of order 2. T is necessarily mapped to an element of order n. There are three possibilities for S.

If S is mapped to an element of order 2, then using the Riemann-Hurwitz formula we can show that there is no normal subgroup that can be obtained in this way. If S is mapped to an element of order 3, then by the Riemann-Hurwitz formula $g = \frac{n-2}{4}$ and therefore the level n of N must be an odd multiple of 2, *i.e.* if $n = 2$, then $N = S_3(\sqrt{3}) \cong (0;2,2,\infty)$, and if $n = 6$, then $N = [3,6]_{1,1}$ with genus 1 (for the notation, see [2]). However for $10 \leq n \leq 26$, there is no normal subgroup of $H(\sqrt{3})$ with three parabolic classes.

Finally if S is mapped to an element of order 6, then similarly $g = \frac{n-1}{2}$ implying that if there exists such a normal subgroup, it must have odd level n.

However as S is mapped to an element of order 6, the normal subgroup (if there is one) must be free and also $6 \mid \mu$. As $\mu = 3n$ with necessarily odd n, this is a contradiction. Therefore we have proved

Theorem 3 *$H(\sqrt{3})$ has no normal free subgroup with three parabolic classes.*

iv. $t = 6$: Here a similar argument gives only two possibilities (except finitely many cyclic quotients);

Firstly, if S is mapped to an element of order 3, then

$$g = \frac{n}{2} - 2$$

implying that such a normal subgroup (if any) must have even level $n \geq 4$. Some examples of these subgroups are obtained for $n = 4, 8$ and 12 as $T_2(\sqrt{3}) \cong [3,4]$, $[3,8]$ and $[3,12]$, respectively. But for the other values of $n \leq 18$, no such normal subgroup exists.

Secondly, if S is mapped to an element of order 6, then

$$g = n - 2$$

and therefore if there exists a free normal subgroup of $H(\sqrt{3})$ having six parabolic classes, then it must be of level $n \geq 2$. There are many examples to this kind of normal subgroups of $H(\sqrt{3})$ with small indices. Actually for $n = 2,3,4,6,8$ and 9, there is a normal subgroup of $H(\sqrt{3})$ with six parabolic classes. But for $n = 5$ and 7, there is no such subgroup. Therefore, although it seems likely that there exist infinitely many normal free subgroup of $H(\sqrt{3})$ having six parabolic classes, it is not clear that this is the case.

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