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ORIGINAL ARTICLE

A subclass with bi-univalence involving (p,q)- Lucas polynomials and its coefficient bounds

S. Yalçın[1] · K. Muthunagai[2] · G. Saravanan[3,4]

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Abstract
We have constructed a subclass of analytic bi-univalent functions using (p,q)-Lucas polynomials in this research contribution. Bounds for certain coefficients and Fekete–Szegö inequalities have been estimated.

Keywords Analytic functions · Bi-univalent functions · $(\mathfrak{p}, \mathfrak{q})$-Lucas polynomials · Fekete-Szegö Inequality

Mathematics Subject Classification 30C45 · 30C15

1 Introduction

Let $\mathfrak{f}(z)$ be a normalized analytic function of the form:

$$\mathfrak{f}(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in U \tag{1.1}$$

where $U = \{z : z \in \mathbb{C}, |z| < 1\}$ and let A be the class of all such functions. Let $S = \{\mathfrak{f}(z) \in A \ni \mathfrak{f}(z)$ is univalent in $U\}$.

✉ K. Muthunagai
muthunagai@vit.ac.in

S. Yalçın
syalcin@uludag.edu.tr

G. Saravanan
gsaran825@yahoo.com

[1] Department of Mathematics, Bursa Uludag University, 16059 Bursa, Turkey

[2] School of Advanced Sciences, VIT University, Chennai 600 127, Tamil Nadu, India

[3] Research Scholar, School of Advanced Sciences, VIT University, Chennai 600 127, Tamil Nadu, India

[4] Present Address: Department of Mathematics, Patrician College of Arts and Science, Adyar, Chennai 600 020, Tamil Nadu, India

For $\mathfrak{f}_1(z)$ and $\mathfrak{f}_2(z) \in A$, we say that $\mathfrak{f}_1(z)$ is subordinate to $\mathfrak{f}_2(z)$, if there exists a function namely, Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ in U, such that $\mathfrak{f}_1(z) = \mathfrak{f}_2(w(z))$, and we write $\mathfrak{f}_1(z) \prec \mathfrak{f}_2(z)$.

Obtaining sharp bounds for $|a_3 - \eta a_2^2|$ of any compact family of functions is called the Fekete–Szegö problem. In particular, when $\eta = 1$, the functional represents Schwarzian derivative and the role of Schwarzian derivative in the theory of Geometric functions is remarkable.

Let Σ denote the class of all bi-univalent functions in U. We say that a function $\mathfrak{f}(z)$ in S belongs to Σ, if both $\mathfrak{f}(z)$ and its inverse have an analytic continuation to $|w| < 1$.

Lewin [9] introduced the class of bi-univalent functions in 1967 and gave an estimate for the second coefficient for functions belonging to this class as $|a_2| < 1.51$. His result was improved by Brannan and Clunie [2] to $|a_2| \leq \sqrt{2}$. There is an extensive literature on the estimates of the initial coefficients of bi-univalent functions (see [3, 6, 7, 14, 19, 20]).

Special polynomials like Lucas polynomials and their generalizations play a vital role in Numerical Analysis, Number Theory, Combinatorics, etc. These polynomials have attracted the attention of several researchers and research has been productive in this field (see, for example, [1, 4, 5, 10–12, 15–18]).

Definition 1.1 (*see* [1, 8]) Consider two polynomials $\mathfrak{p}(x)$ and $\mathfrak{q}(x)$ with real coefficients. The $(\mathfrak{p}, \mathfrak{q})$-Lucas polynomial $\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},n}(\mathfrak{x})$ is given by the recurrence relation:

$$\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},n}(x) = \mathfrak{p}(x)\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},n-1}(x) + \mathfrak{q}(x)\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},n-2}(x) \quad (n \geq 2).$$

The first few Lucas polynomials are listed below:

$$\begin{aligned} &\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},0}(x) = 2, \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},1}(x) = \mathfrak{p}(x), \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},2}(x) \\ &\quad = \mathfrak{p}^2(x) + 2\mathfrak{q}(x), \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},3}(x) = \mathfrak{p}^3(x) + 3\mathfrak{p}(x)\mathfrak{q}(x), \ldots \end{aligned} \tag{1.2}$$

For special cases of $\mathfrak{p}(x)$ and $\mathfrak{q}(x)$, we can get various other polynomials.

If $\mathfrak{p}(x) = x$ and $\mathfrak{q}(x) = 1$, then the $(\mathfrak{p}, \mathfrak{q})$-Lucas polynomial becomes Lucas polynomial $L_n(x)$. By replacing $\mathfrak{p}(x)$ by $2x$ and $\mathfrak{q}(x)$ by 1, respectively, we get Pell–Lucas polynomials $D_n(x)$. For $\mathfrak{p}(x) = 1$ and $\mathfrak{q}(x) = 2x$, the polynomial which we find as a result is Jacobsthal–Lucas polynomial $j_n(x)$. We obtain Fermat–Lucas polynomials $f_n(x)$ when $\mathfrak{p}(x)$ is replaced by $3x$ and $\mathfrak{q}(x)$ by -2. Similarly, Chebyshev polynomials of first kind $T_n(x)$ can be found by choosing $\mathfrak{p}(x)$ as $2x$ and $\mathfrak{q}(x)$ as -1.

Theorem 1.1 (*see* [1, 8]) *Let* $\mathfrak{G}_{\mathfrak{Q}_n(x)}(z)$ *be the generating function of the* $(\mathfrak{p}, \mathfrak{q})$-*Lucas polynomial sequence* $\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},n}(x)$. *Then*:

$$\mathfrak{G}_{\mathfrak{Q}_n(x)}(z) = \sum_{n=0}^{\infty} \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},n}(x) z^n = \frac{2 - \mathfrak{p}(x)z}{1 - \mathfrak{p}(x)z - \mathfrak{q}(x)z^2}.$$

Definition 1.2 A function $\mathfrak{f} \in \Sigma$ is said to be in the class:

$$\mathfrak{A}_\Sigma(\varsigma, t; x) \quad (\varsigma \geq 0; 0 \leq t \leq 1; z, w \in U)$$

if

$$(1-\varsigma)(1-t)\frac{\tilde{\mathfrak{f}}(z)}{z} + (t + \varsigma(1+t))\tilde{\mathfrak{f}}'(z) + \varsigma t(z\tilde{\mathfrak{f}}''(z) - 2) \prec -1 + \mathfrak{G}_{\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},n}(x)}(z),$$

and

$$(1-\varsigma)(1-t)\frac{\mathfrak{g}(w)}{w} + (t + \varsigma(1+t))\mathfrak{g}'(w) + \varsigma t(w\mathfrak{g}''(w) - 2) \prec -1 + \mathfrak{G}_{\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},n}(x)}(w),$$

where $\mathfrak{g} = \tilde{\mathfrak{f}}^{-1}$.

By choosing the parameters ς and t appropriately, the class reduces to known subclasses of bi-univalent functions.

Example 1.1 For $t = 0$ and $\varsigma \geq 1$, $\tilde{\mathfrak{f}} \in \Sigma$ is in the class:

$$\mathfrak{A}_\Sigma(\varsigma, 0; x) = \mathfrak{D}_\Sigma(\varsigma; x) \quad (z, w \in U)$$

if

$$(1-\varsigma)\frac{\tilde{\mathfrak{f}}(z)}{z} + \varsigma\tilde{\mathfrak{f}}'(z) \prec -1 + \mathfrak{G}_{\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},n}(x)}(z),$$

and

$$(1-\varsigma)\frac{\mathfrak{g}(w)}{w} + \varsigma\mathfrak{g}'(w) \prec -1 + \mathfrak{G}_{\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},n}(x)}(w),$$

where $\mathfrak{g} = \tilde{\mathfrak{f}}^{-1}$.

Example 1.2 For $t = 0$ and $\varsigma = 1$, $\tilde{\mathfrak{f}} \in \Sigma$ is in the class:

$$\mathfrak{A}_\Sigma(1, 0; x) = \mathfrak{D}_\Sigma(1; x) = \mathfrak{H}_\Sigma(x) \quad (z, w \in U)$$

if

$$\tilde{\mathfrak{f}}'(z) \prec -1 + \mathfrak{G}_{\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},n}(x)}(z),$$

and

$$\mathfrak{g}'(w) \prec -1 + \mathfrak{G}_{\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},n}(x)}(w),$$

where $\mathfrak{g} = \tilde{\mathfrak{f}}^{-1}$.

2 Coefficient bounds

Theorem 2.1 *Let* $\mathfrak{f}$ *represented by* (1.1) *belong to the class* $\mathfrak{A}_{\Sigma}(\varsigma,t;x)$. *Then*:

$$|a_2|\leq\frac{|\mathfrak{p}(x)|\sqrt{|\mathfrak{p}(x)|}}{\sqrt{|\Big((\varsigma+t)^2+25\varsigma^2t^2+10(\varsigma+t)\varsigma t\Big)\mathfrak{p}^2(x)+2\Big(1+(\varsigma+t)^2+25\varsigma^2t^2+2(\varsigma+t)+10\varsigma t+10(\varsigma+t)\varsigma t\Big)\mathfrak{q}(x)|}}$$

and

$$|a_3|\leq\frac{\mathfrak{p}^2(x)}{(1+(\varsigma+t)+5\varsigma t)^2}+\frac{|\mathfrak{p}(x)|}{1+2(\varsigma+t)+10\varsigma t}.$$

Proof Consider $\mathfrak{f}\in\mathfrak{A}_{\Sigma}(\varsigma,t;x)$. According to Definition 1.2, there exist two analytic functions Θ and Υ with $\Theta(0)=\Upsilon(0)=0$, $|\Theta(z)|<1,|\Upsilon(w)|<1$ for all $z,w\in U$, such that:

$$(1-\varsigma)(1-t)\frac{\mathfrak{f}(z)}{z}+(t+\varsigma(1+t))\mathfrak{f}'(z)+\varsigma t(z\mathfrak{f}''(z)-2)=\mathfrak{G}_{\mathfrak{L}_{\mathfrak{p},\mathfrak{q},n}(x)}(\Theta(z))-1,$$

and

$$(1-\varsigma)(1-t)\frac{\mathfrak{g}(w)}{w}+(t+\varsigma(1+t))\mathfrak{g}'(w)+\varsigma t(w\mathfrak{g}''(w)-2)=\mathfrak{G}_{\mathfrak{L}_{\mathfrak{p},\mathfrak{q},n}(x)}(\Upsilon(w))-1,$$

or equivalently:

$$\begin{aligned}(1-\varsigma)(1-t)\frac{\mathfrak{f}(z)}{z}+(t+\varsigma(1+t))\mathfrak{f}'(z)+\varsigma t(z\mathfrak{f}''(z)-2)&=-1+\mathfrak{L}_{\mathfrak{p},\mathfrak{q},0}(x)\\&+\mathfrak{L}_{\mathfrak{p},\mathfrak{q},1}(x)\Theta(z)+\mathfrak{L}_{\mathfrak{p},\mathfrak{q},2}(x)\Theta^2(z)+\cdots\end{aligned}\tag{2.1}$$

$$\begin{aligned}(1-\varsigma)(1-t)\frac{\mathfrak{g}(w)}{w}+(t+\varsigma(1+t))\mathfrak{g}'(w)+\varsigma t(w\mathfrak{g}''(w)-2)&=-1+\mathfrak{L}_{\mathfrak{p},\mathfrak{q},0}(x)\\&+\mathfrak{L}_{\mathfrak{p},\mathfrak{q},1}(x)\Upsilon(w)+\mathfrak{L}_{\mathfrak{p},\mathfrak{q},2}(x)\Upsilon^2(w)+\cdots\end{aligned}\tag{2.2}$$

(2.1) and (2.2), yield:

$$\begin{aligned}&(1-\varsigma)(1-t)\frac{\mathfrak{f}(z)}{z}+(t+\varsigma(1+t))\mathfrak{f}'(z)+\varsigma t(z\mathfrak{f}''(z)-2)\\&\quad=1+\mathfrak{L}_{\mathfrak{p},\mathfrak{q},1}(x)u_1z+\left[\mathfrak{L}_{\mathfrak{p},\mathfrak{q},1}(x)u_2+\mathfrak{L}_{\mathfrak{p},\mathfrak{q},2}(x)u_1^2\right]z^2+\cdots\end{aligned}\tag{2.3}$$

$$\begin{aligned}&(1-\varsigma)(1-t)\frac{\mathfrak{g}(w)}{w}+(t+\varsigma(1+t))\mathfrak{g}'(w)+\varsigma t(w\mathfrak{g}''(w)-2)\\&\quad=1+\mathfrak{L}_{\mathfrak{p},\mathfrak{q},1}(x)v_1w+\left[\mathfrak{L}_{\mathfrak{p},\mathfrak{q},1}(x)v_2+\mathfrak{L}_{\mathfrak{p},\mathfrak{q},2}(x)v_1^2\right]w^2+\cdots\end{aligned}\tag{2.4}$$

From the fact that if:

$$|\Theta(z)| = |u_1 z + u_2 z^2 + u_3 z^3 + \cdots| < 1 \quad (z \in U)$$

and

$$|\Upsilon(w)| = |v_1 w + v_2 w^2 + v_3 w^3 + \cdots| < 1 \quad (w \in U),$$

then:

$$\begin{aligned} |u_k| &\leq 1, \\ |v_k| &\leq 1 \quad (k \in \mathbb{N}). \end{aligned}$$

Comparing the corresponding coefficients in (2.3) and (2.4), we have:

$$(1 + (\varsigma + t) + 5\varsigma t)a_2 = \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},1}(x)u_1 \tag{2.5}$$

$$(1 + 2(\varsigma + t) + 10\varsigma t)a_3 = \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},1}(x)u_2 + \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},2}(x)u_1^2 \tag{2.6}$$

$$-(1 + (\varsigma + t) + 5\varsigma t)a_2 = \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},1}(x)v_1 \tag{2.7}$$

and

$$\begin{aligned} (1 + 2(\varsigma + t) + 10\varsigma t)(2a_2^2 - a_3) &= \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},1}(x)v_2 \\ &+ \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},2}(x)v_1^2. \end{aligned} \tag{2.8}$$

From Eqs. (2.5) and (2.7), we can easily see that:

$$u_1 = -v_1 \tag{2.9}$$

$$2(1 + (\varsigma + t) + 5\varsigma t)^2 a_2^2 = \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},1}^2(x)(u_1^2 + v_1^2). \tag{2.10}$$

If we add (2.6) to (2.8), we get:

$$2(1 + 2(\varsigma + t) + 10\varsigma t)a_2^2 = \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},1}(x)(u_2 + v_2) + \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},2}(x)(u_1^2 + v_1^2). \tag{2.11}$$

Using (2.10) in the equality (2.11), we have:

$$\begin{aligned} 2\Big((1 + 2(\varsigma + t) + 10\varsigma t)\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},1}^2(x) \\ -(1 + (\varsigma + t) + 5\varsigma t)^2 \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},2}(x)\Big)a_2^2 = \mathfrak{Q}_{\mathfrak{p},\mathfrak{q},1}^3(x)(u_2 + v_2), \end{aligned} \tag{2.12}$$

which implies that:

$$|a_2| \leq \frac{|\mathfrak{p}(x)|\sqrt{|\mathfrak{p}(x)|}}{\sqrt{|\Big((\varsigma + t)^2 + 25\varsigma^2 t^2 + 10(\varsigma + t)\varsigma t\Big)\mathfrak{p}^2(x) + 2\Big(1 + (\varsigma + t)^2 + 25\varsigma^2 t^2 + 2(\varsigma + t) + 10\varsigma t + 10(\varsigma + t)\varsigma t\Big)\mathfrak{q}(x)|}}.$$

Subtracting (2.8) from (2.6), we obtain:

$$2(1+2(\varsigma+t)+10\varsigma t)(a_3-a_2^2)=\mathfrak{L}_{\mathfrak{p},\mathfrak{q},1}(x)(u_2-v_2)+\mathfrak{L}_{\mathfrak{p},\mathfrak{q},2}(x)(u_1^2-v_1^2). \tag{2.13}$$

Then, by virtue of (2.9) and (2.10), Eq. (2.13) becomes:

$$a_3=\frac{\mathfrak{L}_{\mathfrak{p},\mathfrak{q},1}^2(x)(u_1^2+v_1^2)}{2(1+(\varsigma+t)+5\varsigma t)^2}+\frac{\mathfrak{L}_{\mathfrak{p},\mathfrak{q},1}(x)(u_2-v_2)}{2(1+2(\varsigma+t)+10\varsigma t)}.$$

Then, with the help of (1.2), we deduce:

$$|a_3|\leq\frac{\mathfrak{p}^2(x)}{(1+(\varsigma+t)+5\varsigma t)^2}+\frac{|\mathfrak{p}(x)|}{1+2(\varsigma+t)+10\varsigma t}.$$

□

Corollary 2.1 *Let* $\mathfrak{f}\in\mathfrak{D}_\Sigma(\varsigma;x)$. *Then*:

$$|a_2|\leq\frac{|\mathfrak{p}(x)|\sqrt{|\mathfrak{p}(x)|}}{\sqrt{|\varsigma^2\mathfrak{p}^2(x)+2(1+\varsigma)^2\mathfrak{q}(x)|}}.$$

$$|a_3|\leq\frac{\mathfrak{p}^2(x)}{(1+\varsigma)^2}+\frac{|\mathfrak{p}(x)|}{1+2\varsigma}.$$

Corollary 2.2 *Let* $\mathfrak{f}\in\mathfrak{H}_\Sigma(x)$. *Then*:

$$|a_2|\leq\frac{|\mathfrak{p}(x)|\sqrt{|\mathfrak{p}(x)|}}{\sqrt{|\mathfrak{p}^2(x)+8\mathfrak{q}(x)|}}$$

$$|a_3|\leq\frac{\mathfrak{p}^2(x)}{4}+\frac{|\mathfrak{p}(x)|}{3}.$$

3 Fekete–Szegö inequality

In this section, for functions belonging to the class $\mathfrak{A}_\Sigma(\varsigma,t;x)$, we have estimated the bounds for the linear functional.

Theorem 3.1 *Let* $\mathfrak{f}\in\mathfrak{A}_\Sigma(\varsigma,t;x)$ *and* $\eta\in\mathbb{R}$. *Then*:

$$|a_3-\eta a_2^2|\leq\begin{cases}\dfrac{|\mathfrak{p}(x)|}{1+2(\varsigma+t)+10\varsigma t}, & |\eta-1|\leq\left|1-\frac{(1+(\varsigma+t)+5\varsigma t)^2(\mathfrak{p}^2(x)+2\mathfrak{q}(x))}{(1+2(\varsigma+t)+10\varsigma t)\mathfrak{p}^2(x)}\right|.\\ \dfrac{|\mathfrak{p}^3(x)||\eta-1|}{\left|\left((1+\varsigma)^2+25\varsigma^2t^2+10(\varsigma+t)\varsigma t\right)\mathfrak{p}^2(x)+2\mathfrak{q}(x)(1+(\varsigma+t)+5\varsigma t)^2\right|}, & |\eta-1|\geq\left|1-\frac{(1+(\varsigma+t)+5\varsigma t)^2(\mathfrak{p}^2(x)+2\mathfrak{q}(x))}{(1+2(\varsigma+t)+10\varsigma t)\mathfrak{p}^2(x)}\right|.\end{cases}$$

Proof From (2.12) and (2.13):

$$a_3-\eta a_2^2=\frac{\mathfrak{Q}^3_{\mathfrak{p},\mathfrak{q},1}(x)(1-\eta)(u_2+v_2)}{2\Big[(1+2(\varsigma+t)+10\varsigma t)\mathfrak{Q}^2_{\mathfrak{p},\mathfrak{q},1}(x)-(1+(\varsigma+t)+5\varsigma t)^2\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},2}(x)\Big]}+\frac{\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},1}(x)(u_2-v_2)}{2[1+2(\varsigma+t)+10\varsigma t]}$$
$$=\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},1}(x)\left[\left(h(\eta,x)+\frac{1}{2[1+2(\varsigma+t)+10\varsigma t]}\right)u_2+\left(h(\eta,x)-\frac{1}{2[1+2(\varsigma+t)+10\varsigma t]}\right)v_2\right],$$

where

$$h(\eta,x)=\frac{\mathfrak{Q}^2_{\mathfrak{p},\mathfrak{q},1}(x)(1-\eta)}{2\Big[(1+2(\varsigma+t)+10\varsigma t)\mathfrak{Q}^2_{\mathfrak{p},\mathfrak{q},1}(x)-(1+(\varsigma+t)+5\varsigma t)^2\mathfrak{Q}_{\mathfrak{p},\mathfrak{q},2}(x)\Big]}.$$

Then, by (1.2), we conclude that:

$$|a_3-\eta a_2^2|\leq\begin{cases}\dfrac{|\mathfrak{p}(x)|}{1+2(\varsigma+t)+10\varsigma t}, & 0\leq|h(\eta,x)|\leq\dfrac{1}{2(1+2(\varsigma+t)+10\varsigma t)}\\ 2|\mathfrak{p}(x)||h(\eta,x)|, & |h(\eta,x)|\geq\dfrac{1}{2(1+2(\varsigma+t)+10\varsigma t)}.\end{cases}$$

□

Corollary 3.1 *Let* $\mathfrak{f}\in\mathfrak{D}_\Sigma(\varsigma;x)$ *and* $\eta\in\mathbb{R}$. *Then*:

$$|a_3-\eta a_2^2|\leq\begin{cases}\dfrac{|\mathfrak{p}(x)|}{1+2\varsigma}, & |\eta-1|\leq\left|1-\frac{(1+\varsigma)^2(\mathfrak{p}^2(x)+2\mathfrak{q}(x))}{(1+2\varsigma)\mathfrak{p}^2(x)}\right|.\\ \frac{|\mathfrak{p}^3(x)||\eta-1|}{|(1+\varsigma)^2\mathfrak{p}^2(x)+2\mathfrak{q}(x)(1+\varsigma)^2|}, & |\eta-1|\geq\left|1-\frac{(1+\varsigma)^2(\mathfrak{p}^2(x)+2\mathfrak{q}(x))}{(1+2\varsigma)\mathfrak{p}^2(x)}\right|.\end{cases}$$

Corollary 3.2 *Let* $\mathfrak{f}\in\mathfrak{H}_\Sigma(\varsigma;x)$ *and* $\eta\in\mathbb{R}$. *Then*:

$$|a_3-\eta a_2^2|\leq\begin{cases}\dfrac{|\mathfrak{p}(x)|}{3}, & |\eta-1|\leq\left|1-\frac{4(\mathfrak{p}^2(x)+2\mathfrak{q}(x))}{3\mathfrak{p}^2(x)}\right|.\\ \frac{|\mathfrak{p}^3(x)||\eta-1|}{4|\mathfrak{p}^2(x)+2\mathfrak{q}(x)|}, & |\eta-1|\geq\left|1-\frac{4(\mathfrak{p}^2(x)+2\mathfrak{q}(x))}{3\mathfrak{p}^2(x)}\right|.\end{cases}$$

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