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ORIGINAL ARTICLE

The (p, q)-Chebyshev polynomial bounds of a general bi-univalent function class

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Abstract
In the present paper, we will define the bi-univalent function class $\mathcal{S}_{\Sigma}^{\eta,\mu}(p, q)$ related to the (p, q)-Chebyshev polynomials. Then we will derive the (p, q)-Chebyshev polynomial bounds for the initial coefficients and determine Fekete–Szegö functional for $f \in \mathcal{S}_{\Sigma}^{\eta,\mu}(p, q)$.

Keywords (p, q)-Chebyshev polynomials · Bi-univalent functions · Subordination

Mathematics Subject Classification 30C45 · 30C50

1 Introduction and motivation

Let $\mathbf{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbf{C}$ be the set of complex numbers and

$$\mathbf{N} := \{1, 2, 3, \ldots\} = \mathbf{N}_0 \setminus \{0\}$$

be the set of positive integers. Let also $E = \{z \in \mathbf{C} : |z| < 1\}$ be open unit disc in $\mathbf{C}$.

For any integer $n \geq 2$ and $0 < q < p \leq 1$, the (p, q)-Chebyshev polynomials of the second kind is defined by the following recurrence relations:

$$U_n(x, s, p, q) = \left(p^n + q^n\right) x U_{n-1}(x, s, p, q) + (pq)^{n-1} s U_{n-2}(x, s, p, q)$$

with the initial values $U_0(x, s, p, q) = 1$ and $U_1(x, s, p, q) = (p + q)x$ and s is a variable.

In the light of this recurrence relation, we will give the following interesting Table 1.

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Table 1 Some special cases of the (p, q)-Chebyshev polynomials of the second kind

x	s	p	q	$U_n(x, s, p, q)$	(p, q)-Chebyshev polynomials of the second kind
$\frac{x}{2}$	s	p	q	$F_n(x, s, p, q)$	(p, q)-Fibonacci polynomials
x	-1	1	1	$U_n(x)$	Second kind of Chebyshev polynomials
$\frac{x}{2}$	1	1	1	$F_{n+1}(x)$	Fibonacci polynomials
$\frac{1}{2}$	1	1	1	F_{n+1}	Fibonacci numbers
x	1	1	1	$P_{n+1}(x)$	Pell polynomials
1	1	1	1	P_{n+1}	Pell numbers
$\frac{1}{2}$	$2y$	1	1	$J_{n+1}(y)$	Jacobsthal polynomials
$\frac{1}{2}$	2	1	1	J_{n+1}	Jacobsthal numbers

These polynomials defined recursively over the integers share numerous interesting properties and have been extensively studied. They have been also found to be topics of interest in many different areas of pure and applied science, please see the papers [6,8,9,12,14–16,19,21] and closely related references therein.

Very recently, Kızılateş et al. [12] defined (p, q)-Chebyshev polynomials of the first and second kinds and derived explicit formulas, generating functions and some interesting properties of these polynomials.

The generating function of the (p, q)-Chebyshev polynomials of the second kind is as follows:

$$\begin{aligned} G_{p,q}(z) &= \frac{1}{1 - xpz\tau_p - xqz\tau_q - spqz^2\tau_{p,q}} \\ &= \sum_{n=0}^{\infty} U_n(x, s, p, q)z^n \quad (z \in E), \end{aligned}$$

where the Fibonacci operator τ_q was introduced in Ref. [17], by $\tau_q f(z) = f(qz)$. Similarly, $\tau_{p,q} f(z) = f(pqz)$.

Let A represent the class of functions f of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots = z + \sum_{n=0}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc E and normalized under the conditions

$$\begin{aligned} f(0) &= 0, \\ f'(0) &= 1. \end{aligned}$$

Moreover, by S we shall represent the class of all functions in A which are univalent in E. An analytic function f is subordinate to an analytic function g, written

$$f(z) \prec g(z), \quad (z \in E)$$

provided there is an analytic function ϕ, defined on E, with

$$\phi(0) = 0 \quad |\phi(z)| < 1, \quad (z \in E)$$

such that

$$f(z) = g(\phi(z)) \quad (z \in E).$$

The *Koebe One Quarter Theorem* [5] provides that the image of E under every univalent function $f \in A$ contains a disc of radius $1/4$. Thus, every univalent function $f \in A$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$ and $f\left(f^{-1}(w)\right) = w$ $\left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right)$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \tag{2}$$

A function $f \in A$ is said to be bi-univalent in E if both f and f^{-1} are univalent in E. Let Σ denote the class of bi-univalent functions defined in the unit disc E. Historically, Lewin [13] investigated the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Ever since then, motivated substantially by the aforementioned pioneering work on this subject by Srivastava et al. [20], authors discussed estimates on the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses (see, for example, [1,3,4,7,10,11,18]).

2 Bi-univalent function class $\mathcal{S}_{\Sigma}^{\eta,\mu}(p, q)$

In this section, we will define the bi-univalent function class $\mathcal{S}_{\Sigma}^{\eta,\mu}(p, q)$ associated with the (p, q)-Chebyshev polynomials. Then we will derive the (p, q)-Chebyshev polynomial bounds for the initial coefficients and determine Fekete–Szegö functional for $f \in \mathcal{S}_{\Sigma}^{\eta,\mu}(p, q)$.

Definition 1 A function $f \in \Sigma$ is said to be in the class

$$\mathcal{S}_{\Sigma}^{\eta,\mu}(p, q) \quad (\eta \geq 0, 0 < \mu \leq 1, z, w \in E)$$

if the following subordinations are satisfied:

$$\frac{1}{2}\left(\frac{z^{1-\eta} f'(z)}{(f(z))^{1-\eta}} + \left(\frac{z^{1-\eta} f'(z)}{(f(z))^{1-\eta}}\right)^{\frac{1}{\mu}}\right) \prec G_{p,q}(z) \tag{3}$$

and

$$\frac{1}{2}\left(\frac{w^{1-\eta} g'(w)}{(g(w))^{1-\eta}} + \left(\frac{w^{1-\eta} g'(w)}{(g(w))^{1-\eta}}\right)^{\frac{1}{\mu}}\right) \prec G_{p,q}(w), \tag{4}$$

where $g = f^{-1}$.

Remark 1 It should be remarked that the class $\mathcal{S}_{\Sigma}^{\eta,\mu}(p,q)$ is a generalization of well-known classes considered earlier. For example,

– For $\eta = 0$, $p = q = 1$ and $s = -1$, the class $\mathcal{S}_{\Sigma}^{\eta,\mu}(p,q)$ reduce to the class $\mathcal{S}_{\Sigma}^{\eta,\mu}(t)$ which was introduced recently by Altınkaya and Yal çın [2].

Theorem 1 *Let the function f given by (1) be in the class $\mathcal{S}_{\Sigma}^{\eta,\mu}(p,q)$. Then*

$$|a_2| \leq \frac{2\mu(p+q)x\sqrt{(p+q)x}}{\sqrt{(1+\eta)\left|\left[(2+\eta)(\mu^2+\mu)+(1+\eta)(1-\mu)\right](p+q)^2x^2-(1+\eta)(1+\mu)^2\left((p^2+q^2)(p+q)x^2+pqs\right)\right|}},$$

$$|a_3| \leq \frac{2\mu(p+q)x}{1+\mu}\left[\frac{1}{2+\eta}+\frac{2\mu(p+q)x}{(1+\eta)^2(1+\mu)}\right]$$

and for any real number σ,

$$\left|a_3-\sigma a_2^2\right| \leq \begin{cases} \frac{2\mu(p+q)x}{(2+\eta)(1+\mu)}, \\ |1-\sigma| \leq \frac{(1+\eta)\left|\left[(2+\eta)(\mu^2+\mu)+(1+\eta)(1-\mu)\right]-(1+\eta)(1+\mu)^2\left(\frac{(p^2+q^2)}{p+q}+\frac{pqs}{(p+q)^2x^2}\right)\right|}{2(2+\eta)\mu(1+\mu)} \\ \frac{4\mu^2|1-\sigma|(p+q)^3x^3}{(1+\eta)\left|\left[(2+\eta)(\mu^2+\mu)+(1+\eta)(1-\mu)\right](p+q)^2x^2-(1+\eta)(1+\mu)^2\left((p^2+q^2)(p+q)x^2+pqs\right)\right|}, \\ |1-\sigma| \geq \frac{(1+\eta)\left|\left[(2+\eta)(\mu^2+\mu)+(1+\eta)(1-\mu)\right]-(1+\eta)(1+\mu)^2\left(\frac{(p^2+q^2)}{p+q}+\frac{pqs}{(p+q)^2x^2}\right)\right|}{2(2+\eta)\mu(1+\mu)} \end{cases}.$$

Proof Suppose that $f \in \mathcal{S}_{\Sigma}^{\eta,\mu}(p,q)$. Thus, from (3) and (4), we can write

$$\frac{1}{2}\left(\frac{z^{1-\eta}f'(z)}{(f(z))^{1-\eta}}+\left(\frac{z^{1-\eta}f'(z)}{(f(z))^{1-\eta}}\right)^{\frac{1}{\mu}}\right) = G_{p,q}(\phi(z)) \tag{5}$$

and for its inverse map $g = f^{-1}$, we have

$$\frac{1}{2}\left(\frac{w^{1-\eta}g'(w)}{(g(w))^{1-\eta}}+\left(\frac{w^{1-\eta}g'(w)}{(g(w))^{1-\eta}}\right)^{\frac{1}{\mu}}\right) = G_{p,q}(\varphi(w)), \tag{6}$$

for some analytic functions ϕ, φ such that $\phi(0) = \varphi(0) = 0$ and $|\phi(z)| < 1$, $|\varphi(w)| < 1$ for all $z, w \in E$. It is fairly well known that if

$$|\phi(z)| = \left|m_1z+m_2z^2+m_3z^3+\cdots\right| < 1 \quad (z \in E)$$
$$|\varphi(w)| = \left|n_1w+n_2w^2+n_3w^3+\cdots\right| < 1 \quad (w \in E),$$

then

$$|m_k| \leq 1, \quad |n_k| \leq 1 \quad (\forall k \in \mathbf{N}). \tag{7}$$

From the equalities (5) and (6), we obtain that

$$\begin{aligned}\frac{1}{2}\left(\frac{z^{1-\eta}f'(z)}{(f(z))^{1-\eta}}+\left(\frac{z^{1-\eta}f'(z)}{(f(z))^{1-\eta}}\right)^{\frac{1}{\mu}}\right) &= U_0(x,s,p,q)+U_1(x,s,p,q)\phi(z)\\ &\quad +U_2(x,s,p,q)\phi^2(z)+\cdots,\\ \frac{1}{2}\left(\frac{w^{1-\eta}g'(w)}{(g(w))^{1-\eta}}+\left(\frac{w^{1-\eta}g'(w)}{(g(w))^{1-\eta}}\right)^{\frac{1}{\mu}}\right) &= U_0(x,s,p,q)+U_1(x,s,p,q)\varphi(w)\\ &\quad +U_2(x,s,p,q)\varphi^2(w)+\cdots.\end{aligned}$$

Thus, we write

$$\begin{aligned}&\frac{1}{2}\left(\frac{z^{1-\eta}f'(z)}{(f(z))^{1-\eta}}+\left(\frac{z^{1-\eta}f'(z)}{(f(z))^{1-\eta}}\right)^{\frac{1}{\mu}}\right)\\ &\quad=1+U_1(x,s,p,q)m_1z+\left[U_1(x,s,p,q)m_2+U_2(x,s,p,q)m_1^2\right]z^2+\cdots,\end{aligned} \tag{8}$$

$$\begin{aligned}&\frac{1}{2}\left(\frac{w^{1-\eta}g'(w)}{(g(w))^{1-\eta}}+\left(\frac{w^{1-\eta}g'(w)}{(g(w))^{1-\eta}}\right)^{\frac{1}{\mu}}\right)\\ &\quad=1+U_1(x,s,p,q)n_1w+\left[U_1(x,s,p,q)n_2+U_2(x,s,p,q)n_1^2\right]w^2+\cdots.\end{aligned} \tag{9}$$

Now, equating the coefficients in (8) and (9), we obtain

$$\frac{(1+\eta)(1+\mu)}{2\mu}a_2 = U_1(x,s,p,q)m_1, \tag{10}$$

$$\begin{aligned}&\frac{(2+\eta)(1+\mu)}{4\mu}\left(2a_3+(\eta-1)a_2^2\right)+\frac{(1+\eta)^2(1-\mu)}{4\mu^2}a_2^2\\ &\quad=U_1(x,s,p,q)m_2+U_2(x,s,p,q)m_1^2\end{aligned} \tag{11}$$

and

$$-\frac{(1+\eta)(1+\mu)}{2\mu}a_2 = U_1(x,s,p,q)n_1, \tag{12}$$

$$\begin{aligned}&\frac{(2+\eta)(1+\mu)}{4\mu}\left((3+\eta)a_2^2-2a_3\right)+\frac{(1+\eta)^2(1-\mu)}{4\mu^2}a_2^2\\ &\quad=U_1(x,s,p,q)n_2+U_2(x,s,p,q)n_1^2.\end{aligned} \tag{13}$$

From the Eqs. (10) and (12), we can easily see that

$$m_1 = -n_1, \tag{14}$$

$$\frac{(1+\eta)^2(1+\mu)^2}{2\mu^2}a_2^2 = U_1^2(x,s,p,q)\left(m_1^2+n_1^2\right). \tag{15}$$

If we add (11) to (13), we get

$$\left[\frac{(1+\eta)(2+\eta)(1+\mu)}{2\mu}+\frac{(1+\eta)^2(1-\mu)}{2\mu^2}\right]a_2^2 = U_1(x,s,p,q)(m_2+n_2)+U_2(x,s,p,q)\left(m_1^2+n_1^2\right). \tag{16}$$

Using (15) in equality (16),

$$\begin{aligned}&\frac{(1+\eta)}{2\mu^2}\left[(2+\eta)(\mu^2+\mu)+(1+\eta)(1-\mu)-\frac{U_2(x,s,p,q)}{U_1^2(x,s,p,q)}(1+\eta)(1+\mu)^2\right]a_2^2\\ &= U_1(x,s,p,q)(m_2+n_2).\\ &a_2^2 = \frac{2\mu^2U_1^3(x,s,p,q)(m_2+n_2)}{(1+\eta)\left\{\left[(2+\eta)(\mu^2+\mu)+(1+\eta)(1-\mu)\right]U_1^2(x,s,p,q)-(1+\eta)(1+\mu)^2U_2(x,s,p,q)\right\}}.\end{aligned} \tag{17}$$

It follows from (7) and (17) that

$$|a_2| \le \frac{2\mu(p+q)x\sqrt{(p+q)x}}{\sqrt{(1+\eta)\left|\left[(2+\eta)(\mu^2+\mu)+(1+\eta)(1-\mu)\right](p+q)^2x^2-(1+\eta)(1+\mu)^2\left((p^2+q^2)(p+q)x^2+pqs\right)\right|}}.$$

Next, if we subtract (13) from (11), we get

$$\frac{(2+\eta)(1+\mu)}{\mu}\left(a_3-a_2^2\right) = U_1(x,s,p,q)(m_2-n_2)+U_2(x,s,p,q)\left(m_1^2-n_1^2\right). \tag{18}$$

Then Eqs. (14), (15) and (18) become

$$a_3 = \frac{2\mu^2U_1^2(x,s,p,q)\left(m_1^2+n_1^2\right)}{(1+\eta)^2(1+\mu)^2}+\frac{\mu U_1(x,s,p,q)(m_2-n_2)}{(2+\eta)(1+\mu)}.$$

Therefore, from (7), we find that

$$|a_3| \le \frac{2\mu(p+q)x}{1+\mu}\left[\frac{1}{2+\eta}+\frac{2\mu(p+q)x}{(1+\eta)^2(1+\mu)}\right].$$

Additionally, from (17) and (18)

$$
\begin{aligned}
a_3 - \sigma a_2^2 &= (1-\sigma) \\
&\quad \frac{2\mu^2 U_1^3(x,s,p,q)\,(m_2+n_2)}{(1+\eta)\left[(2+\eta)\,(\mu^2+\mu)+(1+\eta)\,(1-\mu)\right]U_1^2(x,s,p,q)-(1+\eta)\,(1+\mu)^2\,U_2(x,s,p,q)} \\
&\quad + \frac{\mu U_1(x,s,p,q)\,(m_2-n_2)}{(2+\eta)\,(\mu+1)} \\
&= U_1(x,s,p,q)\left[\left(h(\sigma)+\frac{\mu}{(2+\eta)\,(\mu+1)}\right)m_2\right. \\
&\quad \left.+\left(h(\sigma)-\frac{\mu}{(2+\eta)\,(\mu+1)}\right)n_2\right],
\end{aligned}
$$

where

$$
h(\sigma) = \frac{2\mu^2 U_1^2(x,s,p,q)\,(1-\sigma)}{(1+\eta)\left[(2+\eta)\,(\mu^2+\mu)+(1+\eta)\,(1-\mu)\right]U_1^2(x,s,p,q)-(1+\eta)\,(1+\mu)^2\,U_2(x,s,p,q)}.
$$

Finally, we have the desired relation

$$
\left|a_3 - \sigma a_2^2\right| \leq \begin{cases} \frac{2\mu(p+q)x}{(2+\eta)(1+\mu)}, & 0 \leq |h(\sigma)| \leq \frac{\mu}{(2+\eta)(1+\mu)} \\ 2(p+q)x\,|h(\sigma)|, & |h(\sigma)| \geq \frac{\mu}{(2+\eta)(1+\mu)} \end{cases}.
$$

3 Conclusions

Our motivation is to obtain many interesting and fruitful usages of a wide variety of the (p, q)-Chebyshev polynomials in geometric function theory. We introduced and investigated the bi-univalent function class $\mathcal{S}_{\Sigma}^{\eta,\mu}(p, q)$ related to the (p, q)-Chebyshev polynomials. Thus, we obtained second and third Taylor–Maclaurin coefficients of functions in this class. These results were an improvement on the estimates obtained in the recent studies.

The geometric properties of the bi-univalent function class $\mathcal{S}_{\Sigma}^{\eta,\mu}(p, q)$ vary according to the values assigned to the parameters. However, some results for the special cases of the parameters included could be expressed as illustrative examples. In this way, one can introduce and study different subclasses of the function class $\mathcal{S}_{\Sigma}^{\eta,\mu}(p, q)$, which we studied in this paper.

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