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# ON A NEW CLASS OF ANALYTIC FUNCTIONS RELATED TO FRACTIONAL CALCULUS

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**Abstract**

Let $\mathcal{A}$ be the class of analytic functions $f$ with $f(0) = 0$ and $f'(0) = 1$ in the open unit disk $\mathbb{U}$. Applying the fractional derivative $D_z^{\lambda} f$ of order $\lambda$ for $f \in \mathcal{A}$, a new subclass $A(\alpha, \beta; \lambda)$ of $\mathcal{A}$ with $0 \leqq \alpha < 1,\ \beta > 1$ and $0 \leqq \lambda < 1$ is considered. The object of the present paper is to discuss some interesting properties of this class of functions.

**Key words:** analytic function, fractional derivative, fractional integral, coefficient inequality

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**1. Introduction.** Let $\mathcal{A}$ be the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

From among the various definitions of $f \in \mathcal{A}$ for fractional calculus (that is, fractional derivatives and fractional integrals) given in the literature, we have to recall here the following definitions for fractional calculus which were used by Owa ([1,2]) and by Owa and Srivastava [3].

**Definition 1.** The fractional integral of order $\lambda$ for $f \in \mathcal{A}$ is defined by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi \quad (\lambda > 0),$$

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where the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi>0$, and $\Gamma(z)$ is the Gamma function.

**Definition 2.** The fractional derivative of order $\lambda$ for $f\in\mathcal{A}$ is defined by

$$\begin{aligned} D_z^{\lambda}f(z) &= \frac{d}{dz}\left(D_z^{\lambda-1}f(z)\right) \\ &= \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\left(\int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}}d\xi\right) \qquad (0\leqq\lambda<1), \end{aligned}$$

where the multiplicity of $(z-\xi)^{-\lambda}$ is removed as in Definition 1 above.

**Definition 3.** Under the hypotheses of Definition 2, The fractional derivative of order $n+\lambda$ for $f\in\mathcal{A}$ is defined by

$$D_z^{n+\lambda}f(z)=\frac{d^n}{dz^n}\left(D_z^{\lambda}f(z)\right) \qquad (0\leqq\lambda<1),$$

where $n=0,1,2,\ldots$.

In view of the above definitions, we see that, for $f\in\mathcal{A}$,

$$\begin{aligned} D_z^{-\lambda}f(z) &= \frac{1}{\Gamma(2+\lambda)}z^{1+\lambda}+\frac{2!}{\Gamma(3+\lambda)}a_2z^{2+\lambda}+\frac{3!}{\Gamma(4+\lambda)}a_3z^{3+\lambda}+\cdots \\ &= \sum_{k=1}^{\infty}\frac{k!}{\Gamma(k+1+\lambda)}a_kz^{k+\lambda} \qquad (a_1=1,\lambda>0), \end{aligned}$$

$$\begin{aligned} D_z^{\lambda}f(z) &= \frac{1}{\Gamma(2-\lambda)}z^{1-\lambda}+\frac{2!}{\Gamma(3-\lambda)}a_2z^{2-\lambda}+\frac{3!}{\Gamma(4-\lambda)}a_3z^{3-\lambda}+\cdots \\ &= \sum_{k=1}^{\infty}\frac{k!}{\Gamma(k+1-\lambda)}a_kz^{k-\lambda} \qquad (a_1=1,0\leqq\lambda<1) \end{aligned}$$

and

$$\begin{aligned} D_z^{n+\lambda}f(z) &= \frac{d^n}{dz^n}\left(D_z^{\lambda}f(z)\right) \\ &= \sum_{k=1}^{\infty}\frac{k!}{\Gamma(k+1-n-\lambda)}a_kz^{k-n-\lambda} \qquad (a_1=1,0\leqq\lambda<1), \end{aligned}$$

where $n=0,1,2,\ldots$.

Applying the fractional derivative $D_z^{\lambda}f$, we introduce the following subclass $A(\alpha,\beta;\lambda)$ of $\mathcal{A}$.

**Definition 4.** A class $A(\alpha, \beta; \lambda)$ for $f \in \mathcal{A}$ is defined by

$$\alpha < \Re\left(\frac{\Gamma(2-\lambda)D_z^{\lambda} f(z)}{z^{1-\lambda}}\right) < \beta \qquad (z \in \mathbb{U})$$

for some real $\alpha\,(0 \leqq \alpha < 1)$ and $\beta\,(\beta > 1)$, where $0 \leqq \lambda < 1$.

Also, letting $\beta \to \infty$, the class $A(\alpha, \infty; \lambda)$ is given by

$$\Re\left(\frac{\Gamma(2-\lambda)D_z^{\lambda} f(z)}{z^{1-\lambda}}\right) > \alpha \qquad (z \in \mathbb{U}). \tag{1}$$

**Remark 1.** It is interesting to note that the special values of $\alpha$ and $\lambda$ lead the class $A(\alpha, \beta; \lambda)$ to various subclasses, we illustrate the following subclasses:

If we consider $\lambda \to 1$ in (1), then the condition

$$\Re(f'(z)) > \alpha \qquad (z \in \mathbb{U})$$

shows that $f$ is close-to-convex of order $\alpha$ in $\mathbb{U}$.

Making $\alpha = 0$ and $\lambda \to 1$, a function $f \in \mathcal{A}$ satisfying

$$\alpha < \Re(f'(z)) < \frac{2\beta}{\beta+1} \qquad (z \in \mathbb{U})$$

for $\beta > 1$ is studied by Altinkaya et al. [4].

**2. Some properties for the class $A(\alpha, \beta; \lambda)$.** First of all, we need the following lemma due to Miller and Mocanu [5] or due to Sakaguchi [6].

**Lemma 1.** *Let $M(z)$ and $N(z)$ be analytic in $\mathbb{U}$ with $M(0) = N(0) = 0$ and $\alpha$ be real. If $N(z)$ maps $\mathbb{U}$ onto a domain which is starlike with respect to the origin, then*

$$\Re\left(\frac{M'(z)}{N'(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

*implies that*

$$\Re\left(\frac{M(z)}{N(z)}\right) > \alpha \qquad (z \in \mathbb{U}).$$

By means of Lemma 1, we obtain

**Lemma 2.** *Let $M(z)$ and $N(z)$ be analytic in $\mathbb{U}$ with $M(0) = N(0) = 0$. Further let $\alpha$ and $\beta$ be real with $0 \leqq \alpha < 1$ and $\beta > 1$. If $N(z)$ maps $\mathbb{U}$ onto a domain which is starlike with respect to the origin, then*

$$\alpha < \Re\left(\frac{M'(z)}{N'(z)}\right) < \beta \qquad (z \in \mathbb{U}) \tag{2}$$

*implies that*

$$\alpha < \Re\left(\frac{M(z)}{N(z)}\right) < \beta \qquad (z \in \mathbb{U}). \tag{3}$$

**Proof.** The parts of left inequality in (2) and (3) are from Lemma 1. If we define the auxiliary function

$$M_1(z) = \beta N(z) - M(z) \quad (z \in \mathbb{U}),$$

then the assumption (2) implies

$$\Re\left(\frac{M_1'(z)}{N'(z)}\right) = \beta - \Re\left(\frac{M'(z)}{N'(z)}\right) > 0.$$

Therefore, the functions $M_1(z)$ and $N(z)$ satisfy the conditions of Lemma 1 with $\alpha = 0$, which gives

$$\Re\left(\frac{M_1(z)}{N(z)}\right) > 0$$

that is

$$\beta - \Re\left(\frac{M(z)}{N(z)}\right) > 0.$$

In this way the upper estimate in (3) is established. □

With the help of Lemma 2, we have

**Theorem 1.** *If $f \in \mathcal{A}$ satisfies*

$$\frac{\alpha}{\Gamma(2-\lambda)} < \Re\left(\lambda z^{\lambda-1} D_z^{\lambda} f(z) + z^{\lambda} D_z^{1+\lambda} f(z)\right) < \frac{\beta}{\Gamma(2-\lambda)} \quad (z \in \mathbb{U}), \tag{4}$$

*for $0 \leqq \alpha < 1,\ \beta > 1$ and $0 \leqq \lambda < 1$, then $f \in A(\alpha, \beta; \lambda)$.*

**Proof.** Let us consider the functions

$$M(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) = z + \frac{2}{2-\lambda} a_2 z^2 + \cdots$$

and $N(z) = z$. Then $M(z)$ and $N(z)$ are analytic in $\mathbb{U}$ with $M(0) = N(0) = 0$. It follows that $N(z)$ is convex (starlike with respect to the origin) in $\mathbb{U}$ and

$$\begin{aligned} \alpha < \ & \Re\left(\frac{M'(z)}{N'(z)}\right) \\ & = \Gamma(2-\lambda)\Re\left(\lambda z^{\lambda-1} D_z^{\lambda} f(z) + z^{\lambda} D_z^{1+\lambda} f(z)\right) < \beta \end{aligned}$$

by condition (4). Therefore, applying Lemma 2, we have that

$$\alpha \leqq \Re\left(\frac{M(z)}{N(z)}\right) \leqq \Re\left(\frac{\Gamma(2-\lambda) D_z^{\lambda} f(z)}{z^{1-\lambda}}\right) \leqq \beta \quad (z \in \mathbb{U}),$$

which shows that $f \in A(\alpha, \beta; \lambda)$. □

**Remark 2.** Letting $\alpha = 0$ and $\lambda \to 1$ in Theorem 3, we have Theorem 1 by Altınkaya et al. [4].

Next, we derive:

**Theorem 2.** *Let $f$ be in the class $A(\alpha,\beta;\lambda)$ and $F(z)$ be defined by*

$$F(z)=D_z^{-\lambda}(3-\lambda)\left(\frac{D_z^{\lambda-1}f(z)}{z}-\frac{D_z^{\lambda-2}f(z)}{z^2}\right)$$

*for $0\leqq\lambda<1$. Then $F$ is in the class $A(\alpha,\beta;\lambda)$.*

**Proof.** We note that

$$\begin{aligned} D_z^{\lambda}F(z) &= (3-\lambda)\left(\frac{D_z^{\lambda-1}f(z)}{z}-\frac{D_z^{\lambda-2}f(z)}{z^2}\right)\\ &= \frac{1}{\Gamma(2-\lambda)}z^{1-\lambda}+\frac{2(3-\lambda)}{(4-\lambda)\Gamma(3-\lambda)}a_2z^{2-\lambda}+\cdots, \end{aligned}$$

that is

$$F(z)=\frac{1}{\Gamma(2-\lambda)}z+\frac{2(3-\lambda)}{(4-\lambda)\Gamma(3-\lambda)}a_2z^2+\cdots.$$

This means that $\Gamma(2-\lambda)F(z)\in A$.

For such a function $F(z)$, we consider

$$\begin{aligned} \frac{\Gamma(2-\lambda)D_z^{\lambda}\left(\Gamma(2-\lambda)F(z)\right)}{z^{1-\lambda}} &= \frac{\Gamma(2-\lambda)^2D_z^{\lambda}F(z)}{z^{1-\lambda}}\\ &= \frac{(3-\lambda)\,\Gamma(2-\lambda)^2\left(zD_z^{\lambda-1}f(z)-D_z^{\lambda-2}f(z)\right)}{z^{3-\lambda}}. \end{aligned}$$

Letting

$$M(z)=(3-\lambda)\,\Gamma(2-\lambda)^2\left(zD_z^{\lambda-1}f(z)-D_z^{\lambda-2}f(z)\right)$$

and

$$N(z)=z^{3-\lambda},$$

we say that $M(z)$ and $N(z)$ are analytic in $\mathbb{U}$ with $M(0)=N(0)=0$. For such $M(z)$ and $N(z)$, we see that

$$\Re\left(\frac{M'(z)}{N'(z)}\right)=\Re\left(\frac{\Gamma(2-\lambda)^2D_z^{\lambda}f(z)}{z^{1-\lambda}}\right).$$

Therefore, for $f\in A(\alpha,\beta;\lambda)$, we arrive at

$$\Gamma(2-\lambda)\alpha<\Re\left(\frac{M'(z)}{N'(z)}\right)<\Gamma(2-\lambda)\beta\quad(z\in\mathbb{U}),$$

where $\Gamma(2-\lambda)\beta>1$. Using Lemma 2, we deduce that

$$\Gamma(2-\lambda)\alpha<\Re\left(\frac{M(z)}{N(z)}\right)=\Re\left(\frac{\Gamma(2-\lambda)^2D_z^{\lambda}F(z)}{z^{1-\lambda}}\right)<\Gamma(2-\lambda)\beta\quad(z\in\mathbb{U}).$$

This completes the proof of Theorem 2. □

Letting $\lambda\to1$ in Theorems 1 and 2, we have the next corollaries.

**Corollary 1.** *If $f \in \mathcal{A}$ satisfies*

$$\alpha < \Re\left(f'(z) + zf''(z)\right) < \beta \quad (z \in \mathbb{U}),$$

*for some $\alpha\,(0 \leqq \alpha < 1)$ and $\beta\,(\beta > 1)$, then*

$$\alpha < \Re(f'(z)) < \beta \quad (z \in \mathbb{U}).$$

**Corollary 2.** *If $f \in \mathcal{A}$ satisfies*

$$\alpha < \Re(f'(z)) < \beta \quad (z \in \mathbb{U}),$$

*for some $\alpha\,(0 \leqq \alpha < 1)$ and $\beta\,(\beta > 1)$, then*

$$\frac{\alpha}{2} < \Re\left(\frac{f(z)}{z} - \frac{\int_0^z f(t)dt}{z^2}\right) < \frac{\beta}{2} \quad (z \in \mathbb{U}).$$

**3. The class $A(\alpha, \infty; \lambda)$.** Letting $\beta \to \infty$ for $A(\alpha, \beta; \lambda)$, we consider the class $A(\alpha, \infty; \lambda)$. To discuss $f$ concerning $A(\alpha, \infty; \lambda)$, we need the following lemma due to CARATHÉODORY [7].

**Lemma 3.** *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and*

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k.$$

*If $p$ satisfies $\Re p(z) > 0 \quad (z \in \mathbb{U})$, then*

$$|c_k| \leqq 2 \quad (k = 1, 2, 3, \ldots).$$

*The equality is sharp for*

$$p(z) = \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z} \quad (0 \leqq \theta < 2\pi).$$

Applying Lemma 3, we have

**Theorem 3.** *If $f \in A(\alpha, \infty; \lambda)$, then*

$$|a_k| \leqq \frac{2(1-\alpha)\Gamma(k+1-\lambda)}{k!\Gamma(2-\lambda)} \quad (k = 2, 3, \ldots).$$

*The result is sharp for $f$ given by*

$$f(z) = D_z^{-\lambda}\left(\frac{z^{1-\lambda}\left(1 + (1-2\alpha)e^{i\theta}z\right)}{\Gamma(2-\lambda)\left(1 - e^{i\theta}z\right)}\right) \quad (0 \leqq \theta < 2\pi).$$

**Proof.** Note that $f \in A(\alpha, \infty; \lambda)$ satisfies

$$\Re\left(\frac{\Gamma(2-\lambda)D_z^{\lambda}F(z)}{z^{1-\lambda}}\right) > \alpha \quad (z \in \mathbb{U})$$

for $0 \leqq \alpha < 1$. We consider a function $g$ defined by

$$\begin{aligned} g(z) &= \frac{1}{1-\alpha}\left(\frac{\Gamma(2-\lambda)D_z^{\lambda}F(z)}{z^{1-\lambda}} - \alpha\right) \\ &= 1 + \sum_{k=2}^{\infty}\frac{k!\Gamma(2-\lambda)}{(1-\alpha)\Gamma(k+1-\lambda)}a_k z^{k-1}. \end{aligned}$$

Then $g$ is analytic in $\mathbb{U}$ with $g(0) = 1$ and $\Re g(z) > 0 \quad (z \in \mathbb{U})$. Therefore, Lemma 3 shows us that

$$\left|\frac{k!\Gamma(2-\lambda)}{(1-\alpha)\Gamma(k+1-\lambda)}a_k\right| \leqq 2 \quad (k = 2, 3, \ldots).$$

This means that

$$|a_k| \leqq \frac{2(1-\alpha)\Gamma(k+1-\lambda)}{k!\Gamma(2-\lambda)} \quad (k = 2, 3, \ldots). \tag{5}$$

Further, the equality in (5) is satisfied for

$$\begin{aligned} g(z) &= \frac{1}{1-\alpha}\left(\frac{\Gamma(2-\lambda)D_z^{\lambda}F(z)}{z^{1-\lambda}} - \alpha\right) \\ &= \frac{1+e^{i\theta}z}{1-e^{i\theta}z}, \end{aligned}$$

that is,

$$D_z^{\lambda}f(z) = \frac{z^{1-\lambda}\left(1+(1-2\alpha)e^{i\theta}z\right)}{\Gamma(2-\lambda)\left(1-e^{i\theta}z\right)} \quad (0 \leqq \theta < 2\pi).$$

This completes the proof of Theorem 3. □

Letting $\alpha = 0$ in Theorem 3, we see

**Corollary 3.** *If $f \in A(0, \infty; \lambda)$, then*

$$|a_k| \leqq \frac{2\Gamma(k+1-\lambda)}{k!\Gamma(2-\lambda)} \quad (k = 2, 3, \ldots).$$

*The result is sharp for $f$ given by*

$$f(z) = D_z^{-\lambda}\left(\frac{z^{1-\lambda}\left(1+e^{i\theta}z\right)}{\Gamma(2-\lambda)\left(1-e^{i\theta}z\right)}\right) \quad (0 \leqq \theta < 2\pi).$$

Next, we derive

**Theorem 4.** *If $f \in \mathcal{A}$ satisfies*

$$\left|a - b\frac{\Gamma(2-\lambda)D_z^\lambda f(z)}{z^{1-\lambda}}\right| < \left|c + b\frac{\Gamma(2-\lambda)D_z^\lambda f(z)}{z^{1-\lambda}}\right| \quad (z \in \mathbb{U}), \tag{6}$$

*for $0 \leqq \lambda < 1$, then $f \in A(\alpha, \infty; \lambda)$, where $a, b$ and $c$ are positive real numbers such that $a - c \geqq 2b\alpha$.*

**Proof.** Let us suppose that

$$g(z) = \frac{\Gamma(2-\lambda)D_z^\lambda f(z)}{z^{1-\lambda}}.$$

Then condition (6) says that

$$|a - bg(z)|^2 < |c + bg(z)|^2. \tag{7}$$

It follows from (7) that

$$b(a+c)(g(z) + \overline{g(z)}) > a^2 - c^2,$$

that is, that

$$\Re g(z) > \frac{a-c}{2b} \quad (z \in \mathbb{U}).$$

Therefore, $a, b$ and $c$ satisfy $a - c \geqq 2b\alpha$, then we say that $f \in A(\alpha, \infty; \lambda)$. □

Similarly, we obtain

**Theorem 5.** *If $f \in \mathcal{A}$ satisfies*

$$\left|a + b\frac{\Gamma(2-\lambda)D_z^\lambda f(z)}{z^{1-\lambda}}\right| < \left|c - b\frac{\Gamma(2-\lambda)D_z^\lambda f(z)}{z^{1-\lambda}}\right| \quad (z \in \mathbb{U}),$$

*for $0 \leqq \lambda < 1$, then $f \in A(0, \beta; \lambda)$, where $a, b$ and $c$ are positive real numbers such that $c - a \leqq 2b\beta$.*

The proof of Theorem 4 is the same as Theorem 3. Then we omit the proof of the theorem.

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