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Some Properties of Certain Multivalent Harmonic Functions

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Abstract: In this paper, various features of a new class of normalized multivalent harmonic functions in the open unit disk are analyzed, including bounds on coefficients, growth estimations, starlikeness and convexity radii. It is further demonstrated that this class is closed when its members are convoluted. It can also be seen that various previously introduced and investigated classes of multivalent harmonic functions appear as special cases for this class.

Keywords: harmonic; multivalent; starlikeness; convexity; convolution

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1. Introduction

A complex-valued harmonic function f in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ is given by $f(z) = h(z) + \overline{g(z)}$ where $h(z)$ and $g(z)$ are analytic in $\mathbb{E}$. We call h the analytic part and g is co-analytic part of f. A necessary and sufficient condition for $f(z)$ to be locally univalent and sense-preserving in $\mathbb{E}$ is $|g'(z)| < |h'(z)|$ for all $z \in \mathbb{E}$ (see [1]). Let $H(m)$, $m \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ denote the class of functions $f(z)$ of the form

$$f(z) = h(z) + \overline{g(z)} = z^m + \sum_{k=m+1}^{\infty} a_k z^k + \overline{\sum_{k=m+1}^{\infty} b_k z^k} \tag{1}$$

which are harmonic in $\mathbb{E}$. We next denote by $S_{H(m)}$ the subclass of functions $f(z) \in H(m)$ which are m-valent and sense-preserving in $\mathbb{E}$. Then, we say that $f(z) \in S_{H(m)}$ is a m-valently harmonic function in $\mathbb{E}$. The functions in $S_{H(m)}$ are harmonic and sense-preserving in $\mathbb{E}$ if $J_f = |h'|^2 - |g'|^2 > 0$ in $\mathbb{E}$. The class $S_{H(1)} \equiv S_H^0$ of the harmonic univalent and sense-preserving functions was studied in detail by Clunie and Sheil-Small [1]. Furthermore, note that $H(1)$ reduces to the class $\mathcal{A}$ of normalized analytic univalent functions in $\mathbb{E}$, if $g(z) \equiv 0$. Let $\mathcal{K}(m)$ and $\mathcal{S}^*(m)$ be the subclasses of $H(m)$ mapping $\mathbb{E}$ onto convex and starlike domains, respectively.

In [2], Owa et al. investigated the class $\mathcal{A}_m(\gamma, \delta, \lambda; j)$ for $\gamma(\gamma > 0)$, $\delta(\delta > 0)$ and $\lambda(0 \leq \lambda < m![\gamma + (m-j)\delta] / (m-j)!)$ and $z \in \mathbb{E}$, which consisted of analytic functions

$$h(z) = z^m + \sum_{k=m+1}^{\infty} a_k z^k$$

such that

$$\mathrm{Re}\left\{\gamma \frac{h^{(j)}(z)}{z^{m-j}} + \delta \frac{h^{(j+1)}(z)}{z^{m-j-1}}\right\} > \lambda, \text{ where } j = 0, 1, 2, \ldots, m-1.$$

The research described in the present paper is motivated by this work and focuses on analyzing the new class of multivalent harmonic functions that is introduced in the following definition:

Definition 1. *Denote by* $G_{H(m)}(\gamma, \delta, \lambda; j)$ *the class of functions* $f = h + \overline{g} \in H(m)$ *and satisfy*

$$Re\left\{\gamma \frac{h^{(j)}(z)}{z^{m-j}} + \delta \frac{h^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right\} > \left|\gamma \frac{g^{(j)}(z)}{z^{m-j}} + \delta \frac{g^{(j+1)}(z)}{z^{m-j-1}}\right| \ (z \in \mathbb{E}), \tag{2}$$

for some $\gamma(\gamma > 0)$, $\delta(\delta > 0)$ *and* $\lambda(0 \leq \lambda < m![\gamma + (m-j)\delta] / (m-j)!)$, *where* $j = 0, 1, 2, \ldots, m-1$.

Remark 1. *If* $m = 1$ *and* $j = 0$, *then the class* $G_{H(1)}(\gamma, \delta, \lambda; 0) \equiv G_H^0(\gamma, \delta, \lambda)$ *is defined by*

$$Re\left\{\gamma \frac{h(z)}{z} + \delta h'(z) - \lambda\right\} > \left|\gamma \frac{g(z)}{z} + \delta g'(z)\right| \ (z \in \mathbb{E}),$$

for some $\gamma \geq 0$, $\delta > 0$ *and* $0 \leq \lambda < \gamma + \delta \leq 1$. *This class was studied by Çakmak et al. (see [3]).*

It is evident that, for $\gamma > 0$, $\delta \geq 0$, $\alpha > 0$ and $0 \leq \lambda < 1$. Here, the following classes are obtained when special values are given to the variables:

$G_{H(1)}(0, 1, 0; 0) = \{f \in H(1) : \mathrm{Re}[h'(z)] > |g'(z)|, z \in \mathbb{E}\}$([4]),
$G_{H(1)}(0, 1, \lambda; 0) = \{f \in H(1) : \mathrm{Re}[h'(z) - \lambda] > |g'(z)|, \ z \in \mathbb{E}\}$([5,6]),
$G_{H(1)}(1, 0, 0; 0) = \left\{f \in H(1) : \mathrm{Re}\left[\frac{h(z)}{z}\right] > \left|\frac{g(z)}{z}\right|, z \in \mathbb{E}\right\}$([7]),
$G_{H(1)}(1, 0, \lambda; 0) = \left\{f \in H(1) : \mathrm{Re}\left[\frac{h(z)}{z} - \lambda\right] > \left|\frac{g(z)}{z}\right|, \ z \in \mathbb{E}\right\}$([7]),
$G_{H(1)}(1-\alpha, \alpha, 0; 0) = \left\{\begin{array}{c} f \in H(1) : \mathrm{Re}\left[(1-\alpha)\frac{h(z)}{z} + \alpha h'(z)\right] \\ > \left|(1-\alpha)\frac{g(z)}{z} + \alpha g'(z)\right|, \ z \in \mathbb{E} \end{array}\right\}$([8]),
$G_{H(1)}(1, 1, 0; 1) = \left\{\begin{array}{c} f \in H(1) : \mathrm{Re}[h'(z) + zh''(z)] \\ > |g'(z) + zg''(z)|, z \in \mathbb{E} \end{array}\right\}$([9]),
$G_{H(1)}(1, \delta, 0; 1) = \left\{\begin{array}{c} f \in H(1) : \mathrm{Re}[h'(z) + \delta zh''(z)] \\ > |g'(z) + \delta zg''(z)|, z \in \mathbb{E} \end{array}\right\}$([10]),
$G_{H(1)}(1, \delta, \lambda; 1) = \left\{\begin{array}{c} f \in H(1) : \mathrm{Re}[h'(z) + \delta zh''(z) - \lambda] \\ > |g'(z) + \delta zg''(z)|, \ z \in \mathbb{E} \end{array}\right\}$([11]),
for $g(z) \equiv 0$, $G_{H(m)}(\gamma, \delta, \lambda; j) \equiv \mathcal{A}_m(\gamma, \delta, \lambda; j)$ ([2]) and
$G_{H(1)}(0, 1, 0; 0) = \{h \in \mathcal{A} : \mathrm{Re}[h'(z)] > 0, \ z \in \mathbb{E}\}$([12]),
$G_{H(1)}(0, 1, \lambda; 0) = \{h \in \mathcal{A} : \mathrm{Re}[h'(z)] > \lambda, \ z \in \mathbb{E}\}$([13]),
$G_{H(1)}(1, 0, 0; 0) = \left\{h \in \mathcal{A} : \mathrm{Re}\left[\frac{h(z)}{z}\right] > 0, \ z \in \mathbb{E}\right\}$([12]),
$G_{H(1)}(1, 0, \lambda; 0) = \left\{h \in \mathcal{A} : \mathrm{Re}\left[\frac{h(z)}{z}\right] > \lambda, \ z \in \mathbb{E}\right\}$([12]),
$G_{H(1)}(1-\alpha, \alpha, 0; 0) = \left\{h \in \mathcal{A} : \mathrm{Re}\left[(1-\alpha)\frac{h(z)}{z} + \alpha h'(z)\right] > 0, \ z \in \mathbb{E}\right\}$([12]),
$G_{H(1)}(1, 1, 0; 1) = \{h \in \mathcal{A} : \mathrm{Re}[h'(z) + zh''(z)] > 0, \ z \in \mathbb{E}\}$([12,14]),
$G_{H(1)}(1, \delta, 0; 1) = \{h \in \mathcal{A} : \mathrm{Re}[h'(z) + \delta zh''(z)] > 0, z \in \mathbb{E}\}$([12]),
$G_{H(1)}(1, \delta, \lambda; 1) = \{h \in \mathcal{A} : \mathrm{Re}[h'(z) + \delta zh''(z)] > \lambda, \ z \in \mathbb{E}\}$([15]),
$G_{H(1)}(\gamma, \delta, \lambda; 0) = \left\{h \in \mathcal{A} : \mathrm{Re}\left[\gamma \frac{h(z)}{z} + \delta h'(z)\right] > \lambda, \ z \in \mathbb{E}\right\}$([16]).

In order to better understand the importance of this subclass analysis, the important subclasses of the multivalent harmonic functions are mentioned first. The subclasses of multivalent harmonic functions are used in a wide range of mathematical and physical contexts, and each subclass has its own unique properties and applications. By studying these subclasses, researchers can gain insights into the behavior and structure of multivalent harmonic functions, and develop new techniques and tools for analyzing complex systems. Some important subclasses are quasiconformal maps, modular forms, harmonic morphisms,

multidimensional harmonic functions, elliptic functions. For more information about these classes, see [17–25].

In Section 2, the necessary and sufficient conditions are specified for certain functions to belong to the class $G_{H(m)}(\gamma, \delta, \lambda; j)$. For the functions of the class, coefficient bounds and growth predictions are also given. The starlikeness and convexity characteristics of the class $G_{H(m)}(\gamma, \delta, \lambda; j)$ are examined in Section 3. In Section 4, convolution properties involving the functions of the class $G_{H(m)}(\gamma, \delta, \lambda; j)$ are examined.

2. The Sharp Coefficient Estimates and Growth Theorems of $G_{H(m)}(\gamma, \delta, \lambda; j)$

The first theorem of this section provides the necessary and sufficient conditions for the functions to be part of the $G_{H(m)}(\gamma, \delta, \lambda; j)$ class. The following theorems concern coefficient estimations and distortion limits regarding this class.

Theorem 1. *The mapping* $\mathfrak{f} = h + \overline{g} \in G_{H(m)}(\gamma, \delta, \lambda; j)$ *if and only if* $F_\epsilon = h + \epsilon g \in \mathcal{A}_m(\gamma, \delta, \lambda; j)$ *for each* $\epsilon(|\epsilon| = 1)$.

Proof. Suppose that $\mathfrak{f} = h + \overline{g} \in G_{H(m)}(\gamma, \delta, \lambda; j)$. For each $\epsilon(|\epsilon| = 1)$,

$$\begin{aligned}
&\mathrm{Re}\left[\gamma\frac{F_\epsilon^{(j)}(z)}{z^{m-j}} + \delta\frac{F_\epsilon^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right] \\
&= \mathrm{Re}\left[\gamma\frac{h^{(j)}(z) + \epsilon g^{(j)}(z)}{z^{m-j}} + \delta\frac{h^{(j+1)}(z) + \epsilon g^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right] \\
&= \mathrm{Re}\left[\gamma\frac{h^{(j)}(z)}{z^{m-j}} + \delta\frac{h^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right] + \mathrm{Re}\left[\epsilon\left(\gamma\frac{g^{(j)}(z)}{z^{m-j}} + \delta\frac{g^{(j+1)}(z)}{z^{m-j-1}}\right)\right] \\
&> \mathrm{Re}\left[\gamma\frac{h^{(j)}(z)}{z^{m-j}} + \delta\frac{h^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right] - \left|\gamma\frac{g^{(j)}(z)}{z^{m-j}} + \delta\frac{g^{(j+1)}(z)}{z^{m-j-1}}\right| \\
&> 0, \quad z \in \mathbb{E}.
\end{aligned}$$

Thus, $F_\epsilon \in \mathcal{A}_m(\gamma, \delta, \lambda; j)$ for each $\epsilon(|\epsilon| = 1)$. Conversely, let $F_\epsilon = h + \epsilon g \in \mathcal{A}_m(\gamma, \delta, \lambda; j)$ then

$$\mathrm{Re}\left[\gamma\frac{h^{(j)}(z)}{z^{m-j}} + \delta\frac{h^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right] > \mathrm{Re}\left[-\epsilon\left(\gamma\frac{g^{(j)}(z)}{z^{m-j}} + \delta\frac{g^{(j+1)}(z)}{z^{m-j-1}}\right)\right] \ (z \in \mathbb{E}).$$

With the convenient choice of $\epsilon(|\epsilon| = 1)$, we have

$$\mathrm{Re}\left[\gamma\frac{h^{(j)}(z)}{z^{m-j}} + \delta\frac{h^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right] > \left|\gamma\frac{g^{(j)}(z)}{z^{m-j}} + \delta\frac{g^{(j+1)}(z)}{z^{m-j-1}}\right| \ (z \in \mathbb{E}),$$

and hence $\mathfrak{f} \in G_{H(m)}(\gamma, \delta, \lambda; j)$. □

Theorem 2. *Let* $\mathfrak{f} = h + \overline{g} \in G_{H(m)}(\gamma, \delta, \lambda; j)$ *then for* $\mathfrak{k} \geq m + 1$,

$$|b_\mathfrak{k}| \leq \frac{(\mathfrak{k} - j)!(\beta - \lambda)}{\mathfrak{k}![\gamma + \delta(\mathfrak{k} - j)]}, \tag{3}$$

where $\beta = \dfrac{m![\gamma + \delta(m - j)]}{(m - j)!}$. *This result is sharp, and equality holds for the following function*

$$\mathfrak{f}(z) = z^m + \frac{(\mathfrak{k} - j)!(\beta - \lambda)}{\mathfrak{k}![\gamma + \delta(\mathfrak{k} - j)]}\overline{z}^\mathfrak{k}. \tag{4}$$

Proof. Suppose that $\mathfrak{f} = h + \overline{g} \in G_{H(m)}(\gamma, \delta, \lambda; j)$. Using the series representation of $g(re^{i\theta})$, $0 \leq r < 1$ and $\theta \in \mathbb{R}$, we derive

$$\begin{aligned}
& r^{\mathfrak{k}-m}[\gamma + \delta(\mathfrak{k} - j)]\frac{\mathfrak{k}!}{(\mathfrak{k} - j)!}|b_{\mathfrak{k}}| \\
\leq & \frac{1}{2\pi}\int_0^{2\pi}\left|\gamma\frac{g^{(j)}(re^{i\theta})}{(re^{i\theta})^{m-j}} + \delta\frac{g^{(j+1)}(re^{i\theta})}{(re^{i\theta})^{m-j-1}}\right|d\theta \\
< & \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left[\gamma\frac{h^{(j)}(re^{i\theta})}{(re^{i\theta})^{m-j}} + \delta\frac{h^{(j+1)}(re^{i\theta})}{(re^{i\theta})^{m-j-1}} - \lambda\right]d\theta \\
= & \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left[\beta - \lambda + \sum_{\mathfrak{k}=m+1}^{\infty}\frac{\mathfrak{k}!}{(\mathfrak{k} - j)!}[\gamma + \delta(\mathfrak{k} - j)]a_{\mathfrak{k}}r^{\mathfrak{k}-m}e^{i(\mathfrak{k}-m)\theta}\right]d\theta \\
= & \beta - \lambda,
\end{aligned}$$

where $\beta = \dfrac{m![\gamma + \delta(m - j)]}{(m - j)!}$. Allowing $r \to 1^-$, we prove the result (3). Moreover, it can be easily seen that the equality is achieved for the function given in (4). □

Theorem 3. *Let $\mathfrak{f} = h + \overline{g} \in G_{H(m)}(\gamma, \delta, \lambda; j)$. Then, for $\mathfrak{k} \geq m + 1$, we have*

$$(i)\ |a_{\mathfrak{k}}| + |b_{\mathfrak{k}}| \leq 2\frac{(\mathfrak{k} - j)!(\beta - \lambda)}{\mathfrak{k}![\gamma + \delta(\mathfrak{k} - j)]},$$

$$(ii)\ ||a_{\mathfrak{k}}| - |b_{\mathfrak{k}}|| \leq 2\frac{(\mathfrak{k} - j)!(\beta - \lambda)}{\mathfrak{k}![\gamma + \delta(\mathfrak{k} - j)]},$$

$$(iii)\ |a_{\mathfrak{k}}| \leq 2\frac{(\mathfrak{k} - j)!(\beta - \lambda)}{\mathfrak{k}![\gamma + \delta(\mathfrak{k} - j)]},$$

where $\beta = \dfrac{m![\gamma + \delta(m - j)]}{(m - j)!}$. All the results given in this theorem are certain and the equations are provided for the following function

$$\mathfrak{f}(z) = z^m + \sum_{\mathfrak{k}=m+1}^{\infty}\frac{2(\mathfrak{k} - j)!(\beta - \lambda)}{\mathfrak{k}![\gamma + \delta(\mathfrak{k} - j)]}z^{\mathfrak{k}}.$$

Proof. (i) Suppose that $\mathfrak{f} = h + \overline{g} \in G_{H(m)}(\gamma, \delta, \lambda; j)$; then, from Theorem 1, $F_\epsilon = h + \epsilon g \in \mathcal{A}_m(\gamma, \delta, \lambda; j)$ for each $\epsilon(|\epsilon| = 1)$. Thus, for each $|\epsilon| = 1$, we have

$$\mathrm{Re}\left[\gamma\frac{h^{(j)}(z) + \epsilon g^{(j)}(z)}{z^{m-j}} + \delta\frac{h^{(j+1)}(z) + \epsilon g^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right] > 0$$

for $z \in \mathbb{E}$. Therefore, there exists an analytic function of the form $p(z) = 1 + \sum_{\mathfrak{k}=1}^{\infty} p_{\mathfrak{k}}z^{\mathfrak{k}}$ with a positive real part in $\mathbb{E}$, such that

$$\gamma\frac{h^{(j)}(z) + \epsilon g^{(j)}(z)}{z^{m-j}} + \delta\frac{h^{(j+1)}(z) + \epsilon g^{(j+1)}(z)}{z^{m-j-1}} = (\beta - \lambda)p(z) + \lambda, \tag{5}$$

where $\beta = \dfrac{m![\gamma + \delta(m-j)]}{(m-j)!}$. Comparing coefficients on both sides of (5), we obtain

$$\frac{\Bbbk![\gamma + \delta(\Bbbk - j)]}{(\Bbbk - j)!}(a_\Bbbk + \epsilon b_\Bbbk) = (\beta - \lambda)p_{\Bbbk-1} \text{ for } \Bbbk \geq m+1.$$

According to Caratheodory [26], since $|p_\Bbbk| \leq 2$ for $\Bbbk \geq 1$, and $\epsilon(|\epsilon| = 1)$ is arbitrary, the proof of (i) is complete. By following the methods in proof (i), proof (ii), and proof (iii) are obtained. The function

$$\mathfrak{f}(z) = z^m + \sum_{\Bbbk=m+1}^{\infty} \frac{2(\Bbbk - j)!(\beta - \lambda)}{\Bbbk![\gamma + \delta(\Bbbk - j)]} z^\Bbbk$$

provides equations. □

We will now give a sufficient condition for a function to be of class $G_{H(m)}(\gamma, \delta, \lambda; j)$.

Theorem 4. *Let* $\mathfrak{f} = h + \overline{g} \in H(m)$ *and where* $\beta = \dfrac{m![\gamma + \delta(m-j)]}{(m-j)!}$ *with*

$$\sum_{\Bbbk=m+1}^{\infty} \frac{\Bbbk![\gamma + \delta(\Bbbk - j)]}{(\Bbbk - j)!}(|a_\Bbbk| + |b_\Bbbk|) \leq \beta - \lambda, \tag{6}$$

then $f = h + \overline{g} \in G_{H(m)}(\gamma, \delta, \lambda; j)$.

Proof. Suppose that $\mathfrak{f} = h + \overline{g} \in H(m)$. Then, using (6),

$$\begin{aligned}
\operatorname{Re}\left[\gamma\frac{h^{(j)}(z)}{z^{m-j}} + \delta\frac{h^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right] &= \operatorname{Re}\left[\beta - \lambda + \sum_{\Bbbk=m+1}^{\infty} \frac{\Bbbk![\gamma + \delta(\Bbbk - j)]}{(\Bbbk - j)!} a_\Bbbk z^{\Bbbk-m}\right] \\
&> \beta - \lambda - \sum_{\Bbbk=m+1}^{\infty} \frac{\Bbbk![\gamma + \delta(\Bbbk - j)]}{(\Bbbk - j)!}|a_\Bbbk| \\
&\geq \sum_{\Bbbk=m+1}^{\infty} \frac{\Bbbk![\gamma + \delta(\Bbbk - j)]}{(\Bbbk - j)!}|b_\Bbbk| \\
&> \left|\sum_{\Bbbk=m+1}^{\infty} \frac{\Bbbk![\gamma + \delta(\Bbbk - j)]}{(\Bbbk - j)!} b_\Bbbk z^{\Bbbk-m}\right| \\
&= \left|\gamma\frac{g^{(j)}(z)}{z^{m-j}} + \delta\frac{g^{(j+1)}(z)}{z^{m-j-1}}\right|,
\end{aligned}$$

where $\beta = \dfrac{m![\gamma + \delta(m-j)]}{(m-j)!}$. Hence, $\mathfrak{f} \in G_{H(m)}(\gamma, \delta, \lambda; j)$. □

Theorem 5. *Let* $\mathfrak{f} = h + \overline{g} \in G_{H(m)}(\gamma, \delta, \lambda; j)$ *for* $\gamma > 0$, $\delta > 0$ *and* $0 \leq \lambda < \beta$, *where* $\beta = m![\gamma + (m-j)\delta]/(m-j)!$, $j = 0, 1, 2, \ldots, m-1$. *Then*

$$\begin{aligned}
\tfrac{m!}{(m-j)!}|z|^{m-j} + 2(\beta - \lambda)\sum_{\Bbbk=1}^{\infty} \frac{(-1)^\Bbbk |z|^{\Bbbk+m-j}}{\delta(\Bbbk + m - j) + \gamma} &\leq \left|\mathfrak{f}^{(j)}(z)\right|, \\
\left|\mathfrak{f}^{(j)}(z)\right| &\leq \tfrac{m!}{(m-j)!}|z|^{m-j} + 2(\beta - \lambda)\sum_{\Bbbk=1}^{\infty} \frac{|z|^{\Bbbk+m-j}}{\delta(\Bbbk + m - j) + \gamma}.
\end{aligned} \tag{7}$$

Inequalities are sharp for the function

$$\mathfrak{f}^{(j)}(z) = \frac{m!}{(m-j)!}z^{m-j} + \sum_{\Bbbk=1}^{\infty} \frac{2(\beta - \lambda)}{\delta(\Bbbk + m - j) + \gamma} z^{\Bbbk+m-j}.$$

Proof. Let $\mathfrak{f} = h + \overline{g} \in G_{H(m)}(\gamma, \delta, \lambda; j)$. Then, using Theorem 1, $F_\epsilon = h + \epsilon g \in \mathcal{A}_m(\gamma, \delta, \lambda; j)$ for each $\epsilon(|\epsilon| = 1)$. Thus, there exists an analytic function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{E}$, such that

$$\gamma \frac{F_\epsilon^{(j)}(z)}{z^{m-j}} + \delta \frac{F_\epsilon^{(j+1)}(z)}{z^{m-j-1}} = \frac{\beta + (\beta - 2\lambda)\omega(z)}{1 - \omega(z)}, \tag{8}$$

where $\beta = m![\gamma + (m-j)\delta]\,/(m-j)!$. Simplifying (8), we obtain

$$\begin{aligned} z^{\frac{\gamma}{\delta}} F_\epsilon^{(j)}(z) &= \frac{1}{\delta} \int_0^z \xi^{\frac{\gamma}{\delta}+m-j-1} \frac{\beta + (\beta - 2\lambda)\omega(\xi)}{1 - \omega(\xi)} d\xi \\ &= \frac{1}{\delta} \int_0^{|z|} \left(\rho e^{i\varphi}\right)^{\frac{\gamma}{\delta}+m-j-1} \frac{\beta + (\beta - 2\lambda)\omega(\rho e^{i\varphi})}{1 - \omega(\rho e^{i\varphi})} e^{i\varphi} d\rho, \end{aligned}$$

where $\xi = \rho e^{i\varphi}$, $0 \le \rho < 1$ and $\varphi \in \mathbb{R}$. Therefore, using the Schwarz lemma, we have

$$\begin{aligned} \left| z^{\frac{\gamma}{\delta}} F_\epsilon^{(j)}(z) \right| &= \left| \frac{1}{\delta} \int_0^{|z|} \left(\rho e^{i\varphi}\right)^{\frac{\gamma}{\delta}+m-j-1} \frac{\beta + (\beta - 2\lambda)\omega(\rho e^{i\varphi})}{1 - \omega(\rho e^{i\varphi})} e^{i\varphi} d\rho \right| \\ &\le \frac{1}{\delta} \int_0^{|z|} \rho^{\frac{\gamma}{\delta}+m-j-1} \frac{\beta + (\beta - 2\lambda)\rho}{1 - \rho} d\rho, \end{aligned}$$

and

$$\left| F_\epsilon^{(j)}(z) \right| \le \frac{m!}{(m-j)!} |z|^{m-j} + 2(\beta - \lambda) \sum_{\mathfrak{k}=1}^{\infty} \frac{|z|^{\mathfrak{k}+m-j}}{\delta(\mathfrak{k}+m-j) + \gamma}.$$

Similarly, we have

$$\begin{aligned} \left| z^{\frac{\gamma}{\delta}} F_\epsilon^{(j)}(z) \right| &= \left| \frac{1}{\delta} \int_0^{|z|} \left(\rho e^{i\varphi}\right)^{\frac{\gamma}{\delta}+m-j-1} \frac{\beta + (\beta - 2\lambda)\omega(\rho e^{i\varphi})}{1 - \omega(\rho e^{i\varphi})} e^{i\varphi} d\rho \right| \\ &\ge \frac{1}{\delta} \int_0^{|z|} \rho^{\frac{\gamma}{\delta}+m-j-1} \mathrm{Re}\left\{ \frac{\beta + (\beta - 2\lambda)\omega(\rho e^{i\varphi})}{1 - \omega(\rho e^{i\varphi})} \right\} d\rho \\ &\ge \frac{1}{\delta} \int_0^{|z|} \rho^{\frac{\gamma}{\delta}+m-j-1} \frac{\beta - (\beta - 2\lambda)\rho}{1 + \rho} d\rho. \end{aligned}$$

and

$$\left| F_\epsilon^{(j)}(z) \right| \ge \frac{m!}{(m-j)!} |z|^{m-j} + 2(\beta - \lambda) \sum_{\mathfrak{k}=1}^{\infty} \frac{(-1)^{\mathfrak{k}} |z|^{\mathfrak{k}+m-j}}{\delta(\mathfrak{k}+m-j) + \gamma}.$$

Since $\epsilon(|\epsilon| = 1)$ is arbitrary, we have (7). □

3. Geometric Properties of Harmonic Mappings in $G_{H(m)}(\gamma, \delta, \lambda; j)$

In this section, the radius of the m-valently starlikeness and convexity for functions in the class $G_{H(m)}(\gamma, \delta, \lambda; j)$ will be provided.

The two following lemmas are used in order to demonstrate the key findings:

Lemma 1 ([27]). *Let $\mathfrak{f} \in H(m)$. If $\sum_{\mathfrak{k}=m+1}^{\infty} \mathfrak{k}(|a_\mathfrak{k}| + |b_\mathfrak{k}|) \le m$, then $\mathfrak{f}$ is m-valently starlike in $\mathbb{E}$.*

Lemma 2 ([28]). *Let $\mathfrak{f} \in H(m)$. If $\sum_{\mathfrak{k}=m+1}^{\infty} \mathfrak{k}^2(|a_\mathfrak{k}| + |b_\mathfrak{k}|) \le m^2$, then $\mathfrak{f}$ is m-valently convex in $\mathbb{E}$.*

The starlikeness result is first presented in the next theorem.

Theorem 6. *Let $f \in G_{H(m)}(\gamma, \delta, \lambda; j)$ be a sense-preserving harmonic mapping in $\mathbb{E}$. Then, f is m-valently starlike in $|z| < R_*$, where*

$$R_* = \inf_{k \geq m+1} \left(\frac{(k-2)!m^2[\gamma + \delta(k-j)]}{(k-j)!2(\beta-\lambda)k} \right)^{\frac{1}{k-m}}.$$

Proof. Let $f \in G_{H(m)}(\gamma, \delta, \lambda; j)$, and let R, $0 < R < 1$, be fixed. Then

$$f_R(z) = R^{-m} f(Rz) = R^{-m} h(Rz) + R^{-m} \overline{g(Rz)} \in G_{H(m)}(\gamma, \delta, \lambda; j)$$

and

$$f_R(z) = z^m + \sum_{k=m+1}^{\infty} a_k R^{k-m} z^k + \overline{\sum_{k=m+1}^{\infty} b_k R^{k-m} z^k}, \; z \in \mathbb{E}.$$

According to Lemma 1, it suffices to show that

$$\sum_{k=m+1}^{\infty} k(|a_k| + |b_k|) R^{k-m} \leq m$$

for $R < R_*$. Using Theorem 3 (i) gives that

$$\sum_{k=m+1}^{\infty} k(|a_k| + |b_k|) R^{k-m} \leq \sum_{k=m+1}^{\infty} \frac{k(k-j)!2(\beta-\lambda)}{k![\gamma + \delta(k-j)]} R^{k-m}. \tag{9}$$

Furthermore, considering that

$$m = m^2 \sum_{k=m+1}^{\infty} \frac{1}{k(k-1)},$$

we know that the inequality (9) can be written by

$$\sum_{k=m+1}^{\infty} \frac{k(k-j)!2(\beta-\lambda)}{k![\gamma + \delta(k-j)]} R^{k-m} \leq m^2 \sum_{k=m+1}^{\infty} \frac{1}{k(k-1)}.$$

Thus, if

$$R^{k-m} \leq \frac{(k-2)![\gamma + \delta(k-j)]m^2}{(k-j)!2(\beta-\lambda)k}$$

for all $k \geq m+1$, then

$$\sum_{k=m+1}^{\infty} k(|a_k| + |b_k|) R^{k-m} \leq m.$$

Therefore, we obtain

$$R_* = \inf_{k \geq m+1} \left(\frac{(k-2)!m^2[\gamma + \delta(k-j)]}{(k-j)!2(\beta-\lambda)k} \right)^{\frac{1}{k-m}}.$$

□

The radius of convexity for the class $G_{H(m)}(\gamma, \delta, \lambda; j)$ is determined in the next theorem.

Theorem 7. *Let $f \in G_{H(m)}(\gamma, \delta, \lambda; j)$ be a sense-preserving harmonic mapping in $\mathbb{E}$. Then f is m-valently convex in $|z| < R_c$, where*

$$R_c = \inf_{k \geq m+1} \left(\frac{(k-2)!m^3[\gamma + \delta(k-j)]}{(k-j)!2(\beta-\lambda)k^2} \right)^{\frac{1}{k-m}}.$$

Proof. Let $\mathfrak{f} \in G_{H(m)}(\gamma, \delta, \lambda; j)$, and let R, $0 < R < 1$, be fixed. Then

$$\mathfrak{f}_R(z) = R^{-m}\mathfrak{f}(Rz) = R^{-m}h(Rz) + R^{-m}\overline{g(Rz)} \in G_{H(m)}(\gamma, \delta, \lambda; j)$$

and

$$\mathfrak{f}_R(z) = z^m + \sum_{\mathfrak{k}=m+1}^{\infty} a_{\mathfrak{k}} R^{\mathfrak{k}-m} z^{\mathfrak{k}} + \overline{\sum_{\mathfrak{k}=m+1}^{\infty} b_{\mathfrak{k}} R^{\mathfrak{k}-m} z^{\mathfrak{k}}}, \; z \in \mathbb{E}.$$

According to Lemma 2, it suffices to show that

$$\sum_{\mathfrak{k}=m+1}^{\infty} \mathfrak{k}^2(|a_{\mathfrak{k}}| + |b_{\mathfrak{k}}|)R^{\mathfrak{k}-m} \leq m^2$$

for $R < R_c$. Using Theorem 3 (i); gives that

$$\sum_{\mathfrak{k}=m+1}^{\infty} \mathfrak{k}^2(|a_{\mathfrak{k}}| + |b_{\mathfrak{k}}|)R^{\mathfrak{k}-m} \leq \sum_{\mathfrak{k}=m+1}^{\infty} \frac{\mathfrak{k}^2(\mathfrak{k}-j)!2(\beta-\lambda)}{\mathfrak{k}![\gamma+\delta(\mathfrak{k}-j)]} R^{\mathfrak{k}-m}. \tag{10}$$

Furthermore, considering that

$$m^2 = m^3 \sum_{\mathfrak{k}=m+1}^{\infty} \frac{1}{\mathfrak{k}(\mathfrak{k}-1)},$$

we know that the inequality (10) can be written by

$$\sum_{\mathfrak{k}=m+1}^{\infty} \frac{\mathfrak{k}(\mathfrak{k}-j)!2(\beta-\lambda)}{\mathfrak{k}![\gamma+\delta(\mathfrak{k}-j)]} R^{\mathfrak{k}-m} \leq m^3 \sum_{\mathfrak{k}=m+1}^{\infty} \frac{1}{\mathfrak{k}(\mathfrak{k}-1)}.$$

Thus, if

$$R^{\mathfrak{k}-m} \leq \frac{(\mathfrak{k}-2)![\gamma+\delta(\mathfrak{k}-j)]m^3}{(\mathfrak{k}-j)!2(\beta-\lambda)\mathfrak{k}^2}$$

for all $\mathfrak{k} \geq m+1$, then

$$\sum_{\mathfrak{k}=m+1}^{\infty} \mathfrak{k}^2(|a_{\mathfrak{k}}| + |b_{\mathfrak{k}}|)R^{\mathfrak{k}-m} \leq m^2.$$

Therefore, we obtain

$$R_c = \inf_{\mathfrak{k}\geq m+1} \left(\frac{(\mathfrak{k}-2)!m^3[\gamma+\delta(\mathfrak{k}-j)]}{(\mathfrak{k}-j)!2(\beta-\lambda)\mathfrak{k}^2} \right)^{\frac{1}{\mathfrak{k}-m}}.$$

□

4. Convex Combinations and Convolutions

In this section, we show that the class $G_{H(m)}(\gamma, \delta, \lambda; j)$ is closed under convex combinations and convolutions of its members.

Theorem 8. *The class $G_{H(m)}(\gamma, \delta, \lambda; j)$ is closed under convex combinations.*

Proof. Suppose $\mathfrak{f}_t = h_t + \overline{g_t} \in G_{H(m)}(\gamma, \delta, \lambda; j)$ for $t = 1, 2, \ldots, n$ and $\sum_{t=1}^{n} \varkappa_t = 1$ $(0 \leq \varkappa_t \leq 1)$. The convex combination of functions $\mathfrak{f}_t$ $(t = 1, 2, \ldots, n)$ may be written as

$$\mathfrak{f}(z) = \sum_{t=1}^{n} \varkappa_t \mathfrak{f}_t(z) = h(z) + \overline{g(z)},$$

where

$$h(z) = \sum_{t=1}^{n} \varkappa_t h_t(z) \text{ and } g(z) = \sum_{t=1}^{n} \varkappa_t g_t(z).$$

Then, both h and g are analytic in $\mathbb{E}$ with $h(0) = g(0) = h'(0) = g'(0) = \ldots = h^{(m-1)}(0) = g^{(m-1)}(0) = h^{(m)}(0) - m! = g^{(m)}(0) = 0$ and

$$\begin{aligned} \mathrm{Re}\left[\gamma\frac{h^{(j)}(z)}{z^{m-j}} + \delta\frac{h^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right] &= \mathrm{Re}\left[\sum_{t=1}^{n} \varkappa_t\left(\gamma\frac{h^{(j)}(z)}{z^{m-j}} + \delta\frac{h^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right)\right] \\ &> \sum_{t=1}^{n} \varkappa_t\left|\gamma\frac{g^{(j)}(z)}{z^{m-j}} + \delta\frac{g^{(j+1)}(z)}{z^{m-j-1}}\right| \\ &\geq \left|\gamma\frac{g^{(j)}(z)}{z^{m-j}} + \delta\frac{g^{(j+1)}(z)}{z^{m-j-1}}\right| \end{aligned}$$

showing that $\mathfrak{f} \in G_{H(m)}(\gamma, \delta, \lambda; j)$. □

A sequence $\{\vartheta_\Bbbk\}_{\Bbbk=0}^{\infty}$ of non-negative real numbers is said to be a convex null sequence, if $\vartheta_\Bbbk \to 0$ as $\Bbbk \to \infty$, and $\vartheta_0 - \vartheta_1 \geq \vartheta_1 - \vartheta_2 \geq \vartheta_2 - \vartheta_3 \geq \ldots \geq \vartheta_{\Bbbk-1} - \vartheta_\Bbbk \geq \ldots \geq 0$. We shall require the following Lemmas 3 and 4 to prove the results of the convolution.

Lemma 3 ([29]). *If $\{\vartheta_\Bbbk\}_{\Bbbk=0}^{\infty}$ be a convex null sequence, then function*

$$Q(z) = \frac{\vartheta_0}{2} + \sum_{\Bbbk=1}^{\infty} \vartheta_\Bbbk z^\Bbbk$$

is analytic and $Re[Q(z)] > 0$ in $\mathbb{E}$.

Lemma 4 ([14]). *Let the function ϕ be analytic in $\mathbb{E}$ with $\phi(0) = 1$ and $Re[\phi(z)] > 1/2$ in $\mathbb{E}$. Then, for any analytic function, ψ in $\mathbb{E}$, the function $\phi * \psi$ takes the values in the convex hull of the image of $\mathbb{E}$ under ψ.*

Lemma 5. *Let $\psi \in \mathcal{A}_m(\gamma, \delta, \lambda; j)$, then $Re\left[\frac{\psi(z)}{z^m}\right] > \frac{1}{2}$.*

Proof. Suppose that $\psi \in \mathcal{A}_m(\gamma, \delta, \lambda; j)$ will be given by $\psi(z) = z^m + \sum_{\Bbbk=m+1}^{\infty} A_\Bbbk z^\Bbbk$, then

$$\mathrm{Re}\left[\beta + \sum_{\Bbbk=m+1}^{\infty} \frac{\Bbbk![\gamma + \delta(\Bbbk - j)]}{(\Bbbk - j)!} A_\Bbbk z^{\Bbbk-m}\right] > \lambda \quad (z \in \mathbb{E}),$$

which is equivalent to $\mathrm{Re}[\phi(z)] > \frac{1}{2}$ in $\mathbb{E}$, where

$$\phi(z) = 1 + \frac{1}{2(\beta - \lambda)} \sum_{\Bbbk=m+1}^{\infty} \frac{\Bbbk![\gamma + \delta(\Bbbk - j)]}{(\Bbbk - j)!} A_\Bbbk z^{\Bbbk-m}.$$

Now consider a sequence $\{\vartheta_\Bbbk\}_{\Bbbk=0}^{\infty}$ defined by

$$\vartheta_0 = 2 \text{ and } \vartheta_{\Bbbk-1} = 2(\beta - \lambda)\frac{(\Bbbk + m - j)!}{(\Bbbk + m)![\gamma + \delta(\Bbbk + m - j)]} \text{ for } \Bbbk \geq 2.$$

It is easy to see that the sequence $\{\vartheta_\Bbbk\}_{\Bbbk=0}^{\infty}$ is a convex null sequence. Using Lemma 3, this explains the following function

$$Q(z) = 1 + 2(\beta - \lambda) \sum_{\Bbbk=1}^{\infty} \frac{(\Bbbk + m - j)!}{(\Bbbk + m)![\gamma + \delta(\Bbbk + m - j)]} z^\Bbbk$$

is analytic and $\mathrm{Re}[Q(z)] > 0$ in $\mathbb{E}$. Writing

$$\frac{\psi(z)}{z^m} = \phi(z) * \left(1 + 2(\beta-\lambda)\sum_{\Bbbk=1}^{\infty} \frac{(\Bbbk+m-j)!}{(\Bbbk+m)![\gamma+\delta(\Bbbk+m-j)]} z^{\Bbbk}\right),$$

and using Lemma 4 gives that $\mathrm{Re}\left\{\frac{\psi(z)}{z^m}\right\} > \frac{1}{2}$ for $z \in \mathbb{E}$. □

Lemma 6. *Suppose $\psi_t \in \mathcal{A}_m(\gamma,\delta,\lambda;j)$ for $t = 1,2$. Then, $\psi_1 * \psi_2 \in \mathcal{A}_m(\gamma,\delta,\lambda;j)$.*

Proof. Let $\psi_1(z) = z^m + \sum_{\Bbbk=m+1}^{\infty} A_{\Bbbk} z^{\Bbbk}$ and $\psi_2(z) = z^m + \sum_{\Bbbk=m+1}^{\infty} B_{\Bbbk} z^{\Bbbk}$. Then, the convolution of $\psi_1(z)$ and $\psi_2(z)$ is defined by

$$\psi(z) = (\psi_1 * \psi_2)(z) = z^m + \sum_{\Bbbk=m+1}^{\infty} A_{\Bbbk} B_{\Bbbk} z^{\Bbbk}.$$

Since $\frac{\psi^{(j)}(z)}{z^{m-j}} = \frac{\psi_1^{(j)}(z)}{z^{m-j}} * \frac{\psi_2(z)}{z^m}$ and $\frac{\psi^{(j+1)}(z)}{z^{m-j-1}} = \frac{\psi_1^{(j+1)}(z)}{z^{m-j}-1} * \frac{\psi_2(z)}{z^m}$, then we have

$$\frac{1}{\beta-\lambda}\left(\gamma\frac{\psi^{(j)}(z)}{z^{m-j}} + \delta\frac{\psi^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right) = \frac{1}{\beta-\lambda}\left(\gamma\frac{\psi_1^{(j)}(z)}{z^{m-j}} + \delta\frac{\psi_1^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right) * \frac{\psi_2(z)}{z^m}. \quad (11)$$

Since $\psi_1 \in \mathcal{A}_m(\gamma,\delta,\lambda;j)$,

$$\mathrm{Re}\left[\gamma\frac{\psi_1^{(j)}(z)}{z^{m-j}} + \delta\frac{\psi_1^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right] > 0 \ (z \in \mathbb{E})$$

and using Lemma 5, $\mathrm{Re}\left[\frac{\psi_2(z)}{z^m}\right] > \frac{1}{2}$ in $\mathbb{E}$. Now applying Lemma 4 to (11) yields $\mathrm{Re}\left[\gamma\frac{\psi^{(j)}(z)}{z^{m-j}} + \delta\frac{\psi^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right] > 0$ in $\mathbb{E}$. Thus, $\psi = \psi_1 * \psi_2 \in \mathcal{A}_m(\gamma,\delta,\lambda;j)$. □

Now, using Lemma 6, we prove that the class $G_{H(m)}(\gamma,\delta,\lambda;j)$ is closed under the convolutions of its members.

Theorem 9. *Let $\mathfrak{f}_t \in G_{H(m)}(\gamma,\delta,\lambda;j)$ for $t = 1,2$. Then, $\mathfrak{f}_1 * \mathfrak{f}_2 \in G_{H(m)}(\gamma,\delta,\lambda;j)$.*

Proof. Suppose $\mathfrak{f}_t = h_t + \overline{g_t} \in G_{H(m)}(\gamma,\delta,\lambda;j)$ $(t = 1,2)$. Then, the convolution of $\mathfrak{f}_1$ and $\mathfrak{f}_2$ is defined as $\mathfrak{f}_1 * \mathfrak{f}_2 = h_1 * h_2 + \overline{g_1 * g_2}$. In order to show that $\mathfrak{f}_1 * \mathfrak{f}_2 \in G_{H(m)}(\gamma,\delta,\lambda;j)$, we need to show that $F_\epsilon = h_1 * h_2 + \epsilon(g_1 * g_2) \in \mathcal{A}_m(\gamma,\delta,\lambda;j)$ for each $\epsilon(|\epsilon| = 1)$. By Lemma 6, the class $\mathcal{A}_m(\gamma,\delta,\lambda;j)$ is closed under convolutions for each $\epsilon(|\epsilon| = 1)$, $h_t + \epsilon g_t \in \mathcal{A}_m(\gamma,\delta,\lambda;j)$ for $t = 1,2$. Then, both ψ_1 and ψ_2 given by

$$\psi_1 = (h_1 - g_1) * (h_2 - \epsilon g_2) \ \text{ and } \ \psi_2 = (h_1 + g_1) * (h_2 + \epsilon g_2),$$

are members of $\mathcal{A}_m(\gamma,\delta,\lambda;j)$. Since $\mathcal{A}_m(\gamma,\delta,\lambda;j)$ is closed under convex combinations, then the function

$$F_\epsilon = \frac{1}{2}(\psi_1 + \psi_2) = h_1 * h_2 + \epsilon(g_1 * g_2)$$

is a member of $\mathcal{A}_m(\gamma,\delta,\lambda;j)$. Thus, $G_{H(m)}(\gamma,\delta,\lambda;j)$ is closed under convolution. □

In [30], Goodloe defined the Hadamard product of a harmonic function $\mathfrak{f} = h + \overline{g}$ with an analytic function ϕ in $\mathbb{E}$ as follows:

$$\mathfrak{f} \widetilde{*} \phi = h * \phi + \overline{g * \phi}.$$

Theorem 10. *Suppose that* $\mathfrak{f} \in G_{H(m)}(\gamma, \delta, \lambda; j)$ *and* $\phi \in \mathcal{A}(m)$ *with* $Re\left[\frac{\phi(z)}{z^m}\right] > \frac{1}{2}$ *for* $z \in \mathbb{E}$, *then* $\mathfrak{f} \widetilde{*} \phi \in G_{H(m)}(\gamma, \delta, \lambda; j)$.

Proof. Suppose that $\mathfrak{f} = h + \overline{g} \in G_{H(m)}(\gamma, \delta, \lambda; j)$, then $F_\epsilon = h + \epsilon g \in \mathcal{A}_m(\gamma, \delta, \lambda; j)$ for each $\epsilon(|\epsilon| = 1)$. By Theorem 1, in order to prove that $\mathfrak{f} \widetilde{*} \phi \in G_{H(m)}(\gamma, \delta, \lambda; j)$, we need to prove that $\psi = h * \phi + \epsilon(g * \phi) \in \mathcal{A}_m(\gamma, \delta, \lambda; j)$ for each $\epsilon(|\epsilon| = 1)$. Write ψ as $\psi = F_\epsilon * \phi$, and

$$\frac{1}{\beta - \lambda}\left(\gamma \frac{\psi^{(j)}(z)}{z^{m-j}} + \delta \frac{\psi^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right) = \frac{1}{\beta - \lambda}\left(\gamma \frac{F_\epsilon^{(j)}(z)}{z^{m-j}} + \delta \frac{F_\epsilon^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right) * \frac{\phi(z)}{z^m}.$$

Since $Re\left[\frac{\phi(z)}{z^m}\right] > \frac{1}{2}$ and $Re\left[\gamma \frac{F_\epsilon^{(j)}(z)}{z^{m-j}} + \delta \frac{F_\epsilon^{(j+1)}(z)}{z^{m-j-1}} - \lambda\right] > 0$ in $\mathbb{E}$, Lemma 4 yields $\psi \in \mathcal{A}_m(\gamma, \delta, \lambda; j)$. □

5. Discussion and Conclusions

As stated in the information provided in the introduction, multivalent harmonic functions are becoming more and more significant. A new class of these functions is defined in this paper and is denoted by $G_{H(m)}(\gamma, \delta, \lambda; j)$ in Definition 1. By assigning various values to the variables in the class defined in this study, it is established that numerous subclasses that have already been investigated by many different researchers can be obtained. The coefficient relations, growth theorems, and geometric features of the recently introduced class $G_{H(m)}(\gamma, \delta, \lambda; j)$ are examined in the four sections of this research paper.

We believe that, as this article thoroughly covers the topic of multivalent harmonic functions from the past to the present, it will be relevant to numerous future studies. We suggest future studies regarding the theories of differential subordination and superordination involving functions from the class $G_{H(m)}(\gamma, \delta, \lambda; j)$ as seen for harmonic complex-valued functions in recent papers such as [31,32]. Quantum calculus aspects can be introduced in this study, as presented in recent investigations [33–35].

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