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A New Method for the Sum-Edge Characteristic Polynomials of Graphs

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ABSTRACT: In this paper, the determinant of the sum-edge adjacency matrix of any given graph without loops is calculated by means of an algebraic method using spanning elementary subgraphs and also the coefficients of the corresponding sum-edge characteristic polynomial are determined by means of the elementary subgraphs. Also, we provide a formula for calculating the number of smallest odd-sized cycles in a given regular graph.

Key Words: Graphs, Adjacency, Sum-edge adjacency, Characteristic polynomial, Permutation.

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1. Significance of the work

In the last seven decades, graphs have been implemented increasingly to model real life situations to obtain numerical data by mathematical ways which can be commented to obtain physical or chemical information normally obtained as a result of time and money consuming laboratory experiments. There are three main ways of transforming such a case to mathematical language: by means of vertex degrees, matrices or distances. In this work, we give an algebraic method for one of the matrices called the sum-edge characteristic polynomials corresponding to graphs.

2. Introduction

Let G be an arbitrary graph with no loops. A lot of algebraic, physical and chemical properties of a given graph G can be determined by means of matrices corresponding to G. The most famous such matrices are the adjacency, incidency and Laplacian matrices. For example, the intermolecular energy of a chemical compound can be found only by mathematical calculations on the graph modelling the chemical compound as the sum of the absolute values of the eigenvalues of the adjacency matrix, see [4], [5], [7] and [12]. There are some other graph matrices with several uses, see e.g. [2] and [11]. Accordingly, several types of characteristic polynomials and energy are defined by means of these matrix types, [16] and [17]. One of the most popular of them is the Laplacian energy, see e.g. [8]. In [9], some degree based energies have been studied. Das studied maximum energy of bipartite graphs in [10]. In [13], energy of regular graphs is considered. Naturally, being a linear algebraic concept, graph energy is related to many other notions related to graphs. For example the relation between energy and matching number have been considered in [1] and [15]. In this paper, we study the sum-edge adjacency matrix and we determine the sum-edge characteristic polynomial.

3. Sum-edge adjaceny matrices, determinants and characteristic polynomials

Firstly, we recall the definition of sum-edge adjacency matrix of G, see e.g. [11].

Let the vertex set of G be $V(G) = \{v_1, \cdots, v_n\}$ and edge set $E(G)$. We shall denote the sum-edge adjacency matrix of G by $S(G)$. $\mathrm{S(G)} = [s_{ij}]_{n\times n}$ is determined by the adjacency of vertices as follows:

$$s_{ij} = \begin{cases} d(v_i) + d(v_j), & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

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$S(G)$ is a symmetric $n \times n$ matrix.

We will call the characteristic polynomial of $S(G)$ as the sum-edge characteristic polynomial of G and denote it by $P_G^{se}(x)$. These polynomials for several graphs were studied in [14].

Secondly, let us take a graph G. A subgraph G'_t of G is called an elementary subgraph if every component of G'_t is either an edge or a cycle, see e.g. [2]. Also, if the number of vertices of an elementary subgraph is equal to the number of vertices of G, then the elementary subgraph is called a spanning elementary subgraph. $c_-(G'_t)$ and $c_\circ(G'_t)$ are defined as the number of components in a subgraph G'_t which are edges and cycles, respectively.

Definition 3.1. *[6] Let $M = [m_{ij}]_{n\times n}$ be a matrix. The determinant function of M which will be denoted by $|M|$, is defined by*

$$|M| = \sum (^{+}_{-}) m_{1i_1} m_{2i_2} \cdots m_{ni_n}$$

where the summation is over all permutations $i_1 i_2 \cdots i_n$ of the set $S = \{1, 2, \cdots, n\}$. Since for a set with n element, the number of permutations is given by $n!$, we sum all $n!$ terms. In other words, we can express the formula as follows:

Let $S(G) = [s_{ij}]_{n\times n}$ be the sum-edge adjacency matrix of G. Then

$$|S(G)| = \sum sgn(\sigma) s_{1\sigma(1)} s_{2\sigma(2)} \cdots s_{n\sigma(n)}$$

where the summation is over all permutations of $\{1, \cdots, n\}$. The sign is taken as $+$ or $-$ with respect to whether the permutation is even or odd.

Let us define new notions in preparation to the forthcoming theorems. Let u_i and u_j be two adjacent vertices in an edge component of corresponding elementary (spanning elementary) subgraph G'_t of G. Similarly, let v_i and v_j be two adjacent vertices in a cycle component of corresponding elementary (spanning elementary) subgraph G'_t of G. In the next theorem, we provide a formula for calculating the determinant of $S(G)$ whose proof is completed by using a method given in [3].

Theorem 3.2. *Let G be any graph with vertex set $V(G) = \{v_1, \cdots, v_n\}$ and edge set $E(G)$. Let $S(G)$ be the sum-edge adjacency matrix of G. Then*

$$|S(G)| = \sum (-1)^{n-c_-(G'_t)-c_\circ(G'_t)} 2^{c_\circ(G'_t)} \prod_{\substack{i,j=1 \\ j>i}}^{n} [d(u_i) + d(u_j)]^2 \prod_{\substack{i,j=1 \\ j>i}}^{n} [d(v_i) + d(v_j)]$$

where the summation is over all spanning elementary subgraphs G'_t.

Note that in a cycle component of the spanning elementary subgraph G'_t of G, if v_i and v_j are two adjacent vertices in this cycle component, then when calculating the latter product in Theorem 3.2 corresponding to cycle component, we consider only one of the edges v_iv_j and v_jv_i due to the permutation structure. Let now u_i and u_j be two adjacent vertices in an edge component of the spanning elementary subgraph G'_t of G. We consider both of the edges u_iu_j and u_ju_i due to the transposition structure and therefore the square appears in the first product corresponding to edge components.

If either one of the cycle components or edge components does not exist in the spanning elementary subgraph, we exclude the corresponding product in the statement of Theorem 3.2.

Proof. By the Definition 3.1,

$$|S(G)| = \sum sgn(\sigma) s_{1\sigma(1)} s_{2\sigma(2)} \cdots s_{n\sigma(n)}.$$

If some $s_{i\sigma(i)} = 0$ for $i = 1, \cdots, n$, then this means that v_i and $v_{\sigma(i)}$ are not adjacent vertices. Hence, if the term corresponding to a permutation σ is non-zero, then we know that any permutation can be

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expressed uniquely as the product of disjoint cycles whose length at least two. Every cycle (ij) of length two corresponds to the factors $s_{ij}s_{ji}$ and represents a single edge (v_iv_j) in G. Hence, $s_{ij}s_{ji}$ means that $[d(u_i)+d(u_j)][d(u_j)+d(u_i)]$ is calculated, and is therefore equal to $[d(u_i)+d(u_j)]^2$. If any spanning elementary subgraph has at least two edge components, then each edge component contributes to the product by $s_{ij}s_{ji}$. Moreover, every cycle $(123\cdots n)$ of length greater than or equal to three is in the form $(v_1v_2\cdots v_n)$ and corresponds to the product by $s_{12}s_{23}\cdots s_{n1}$. This means that the contribution of the cycle components to the determinant will be $[d(v_1)+d(v_2)][d(v_2)+d(v_3)]\cdots[d(v_n)+d(v_1)]$. If any spanning elementary subgraph has at least two cycle components, then each such cycle component contributes to the corresponding product by $s_{12}s_{23}\cdots s_{n1}$. Also, any spanning elementary subgraph consists of both cycle and edge components. Then both of these component types contribute factors as we have mentioned above. Thus, as a result, every non-zero term in the determinant expansion is an element of an elementary subgraph G'_t of G.

As we have already noted before, any permutation σ can be expressed in a unique way as the product of some disjoint cycles and in such a representation, the sign of σ will be

$$(-1)^{n-\text{the number of cycles in the cycle decomposition of }\sigma}$$

which is equal to $(-1)^{n-c_-(G'_t)-c_\circ(G'_t)}$.

Now, we consider cycle components in G'_t. For each of these, we have two choices, so such permutations corresponding to cycle components contribute $2^{c_\circ(G'_t)}$. Thus, we finish the proof. □

Example 3.3. *Let us find the determinant of the sum-edge adjacency matrix $S(G)$ of the given tadpole graph $T_{6,2}$ in Fig. 1 by using Theorem 3.2.*

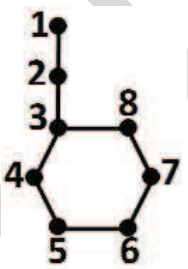


Figure 1: Graph $T_{6,2}$

First, we find all spanning elementary subgraphs of $T_{6,2}$. We have three possibilities:

$E(G'_1)=\{12,34,56,78\}$, $E(G'_2)=\{12,45,67,38\}$ *and* $E(G'_3)=\{345678,12\}$.

Now $E(G'_1)=\{12,34,56,78\}$ *contributes* $(-1)^{8-4-0}2^0\cdot 3^2\cdot 5^2\cdot 4^2\cdot 4^2=57600$ *to the determinant, similarly* $E(G'_2)=\{12,45,67,38\}$ *contributes* $(-1)^{8-4-0}2^0\cdot 3^2\cdot 4^2\cdot 4^2\cdot 5^2=57600$ *to the determinant, and finally* $E(G'_3)=\{345678,12\}=\{34,45,56,67,78,83,12\}$ *contributes to the determinant by* $(-1)^{8-1-1}2^1 5\cdot 4\cdot 4\cdot 4\cdot 4\cdot 5\cdot 3^2=115200$ *and as a result, by Theorem 3.2, we found*

$$|S(G)|=2(57600)+(115200)=230400.$$

We can now give the formula for the factors of the sum-edge characteristic polynomial of any graph G by means of [2]:

Theorem 3.4. *Let G be any graph of order n. Then*

$$P^{se}_G(x)=|xI-S(G)|=x^n+k_1x^{n-1}+k_2x^{n-2}+\cdots+k_n,$$

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where

$$k_l = \sum (-1)^{c_-(G'_t)+c_\circ(G'_t)} 2^{c_\circ(G'_t)} \prod_{\substack{i,j=1 \\ j>i}}^{n} [d(u_i)+d(u_j)]^2 \prod_{\substack{i,j=1 \\ j>i}}^{n} [d(v_i)+d(v_j)].$$

Here the summation is over all elementary subgraphs G'_t with l vertices.

Proof. Every k_l is calculated by means of the corresponding elementary subgraphs of G with l vertices. As a method for calculating value of k_l, we have to calculate all principal minors of $S(G)$ with l vertices. After that, we sum all of these and lastly multiply with $(-1)^l$, for $l = 1, \cdots, n$. By Theorem 3.2, we can write determinant formula instead of principal minors and hence, we achive the formula

$$k_l = (-1)^l \sum (-1)^{l-c_-(G'_t)-c_\circ(G'_t)} 2^{c_\circ(G'_t)} \prod_{\substack{i,j=1 \\ j>i}}^{n} [d(u_i)+d(u_j)]^2 \prod_{\substack{i,j=1 \\ j>i}}^{n} [d(v_i)+d(v_j)]$$

which is equal to the formula

$$k_l = \sum (-1)^{c_-(G'_t)+c_\circ(G'_t)} 2^{c_\circ(G'_t)} \prod_{\substack{i,j=1 \\ j>i}}^{n} [d(u_i)+d(u_j)]^2 \prod_{\substack{i,j=1 \\ j>i}}^{n} [d(v_i)+d(v_j)]$$

where the summation is over all elementary subgraphs G'_t having l vertices. □

Example 3.5. *Let us find the sum-edge characteristic polynomial of the given graph G by the help of Theorem 3.4.*

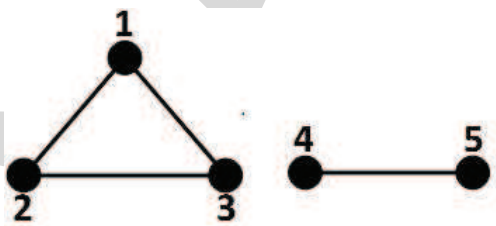


Figure 2: Graph G

Let the sum-edge characteristic polynomial $P^{se}_G(x)$ of G be $P^{se}_G(x) = x^5 + k_1x^4 + k_2x^3 + k_3x^2 + k_4x + k_5$. It is clear that $k_1 = 0$.

For k_2, all elementary subgraphs of G with two vertices are

$$E(G'_1) = \{12\},\ E(G'_2) = \{13\},\ E(G'_3) = \{23\},\ E(G'_4) = \{45\}.$$

Now $E(G'_1) = \{12\}$ contributes $(-1)^1 \cdot 2^0 \cdot 4^2 = -16$ to k_2, $E(G'_2) = \{13\}$ contributes $(-1)^1 \cdot 2^0 \cdot 4^2 = -16$ to k_2, $E(G'_3) = \{23\}$ contributes $(-1)^1 \cdot 2^0 \cdot 4^2 = -16$ to k_2 and $E(G'_4) = \{45\}$ contributes $(-1)^1 \cdot 2^0 \cdot 2^2 = -4$ to k_2. Then adding all these up, we get $k_2 = -52$.

For k_3, the unique elementary subgraph of G with three vertices is

$$E(G'_5) = \{123\} = \{12, 23, 31\}.$$

$E(G'_5)$ contributes $(-1)^1 2^1 4.4.4 = -128$ to k_3. Then $k_3 = -128$.

For k_4, all possible elementary subgraphs of G with four vertices are

$$E(G'_6) = \{12, 45\},\ E(G'_7) = \{13, 45\} \text{ and } E(G'_8) = \{23, 45\}.$$

Here we find that $E(G_6') = \{12, 45\}$ *contributes* $(-1)^2 2^0 4^2 2^2 = 64$, $E(G_7') = \{13, 45\}$ *contributes* $(-1)^2 2^0 4^2 2^2 = 64$ *and* $E(G_8') = \{23, 45\}$ *contributes* $(-1)^2 2^0 4^2 2^2 = 64$ *to* k_4. *Then their sum is equal to* $k_4 = 192$.

For k_5, *the unique elementary subgraph of* G *with five vertices is*

$$E(G_9') = \{123, 45\} = \{12, 23, 31, 45\}.$$

$E(G_9')$ *contibutes* $(-1)^2 2^1 4.4.4.2^2 = 512$ *to* k_5. *That is,* $k_5 = 512$.

Consequently, by Theorem 3.4,

$$P_G^{se}(x) = x^5 - 52x^3 - 128x^2 + 192x + 512.$$

We can now prove the following result by means of [3]:

Theorem 3.6. *Let* G *be a graph of order* n. *Let*

$$P_G^{se}(x) = |xI - S(G)| = x^n + k_1 x^{n-1} + k_2 x^{n-2} + \cdots + k_n.$$

If $k_1, k_3, \cdots, k_{2m+1} = 0$, *then* G *has no odd cycle of length* $2i+1$ *where* $i = 0, 1, \cdots, m$. *Moreover, if* G *is a regular graph, then the number of* $(2m+3)-$*cycles in* G *is*

$$\frac{-k_{2m+3}}{2\prod_{\substack{i,j=1\\ j>i}}^{n}[d(v_i) + d(v_j)]}.$$

Note that v_i *and* v_j *are adjacent vertices in one of the elementary subgraphs with* $2m+3$ *vertices.*

Proof. For any graph G, the first coefficient of the characteristic polynomial of G is $k_1 = 0$. Since we assume that $k_3 = 0$, there are no 3-cycles in G. Hence, we next look for 5-cycles. If G has at least one 5-cycle, we have the result. If $k_5 = 0$, then there are no 5-cycles in G. We continue the process until we find all $k_1, k_3, \cdots, k_{2m+1} = 0$. Finally we have to obtain an elementary graph having at least one k_{2m+3}-cycle. The elementary subgraph will have $2m+3$-cycles. Let one of these cycles be $(1\ 2\ \cdots\ (2m+3))$. Then we have the edge decomposition $v_{12}v_{23}\cdots v_{(2m+3)1}$. By Theorem 3.4, we have

$$k_{2m+3} = \sum (-1)^{c_-(G_t')+c_\circ(G_t')} 2^{c_\circ(G_t')} \prod_{\substack{i,j=1\\ j>i}}^{n} [d(v_i) + d(v_j)]$$

where the summation is over all elementary subgraphs G_t' each of which is a $(2m+3)$-cycle in G. If the graph is regular, each non-zero $[d(v_i)+d(v_j)]$ has the same value and so each $(2m+3)$-cycle component in G_t' contributes the same amount to the k_{2m+3}. Furthermore, each component of the elementary subgraph G_t' is a $(2m+3)$-cycle, we have $c_-(G_t') = 0$ and $c_\circ(G_t') = 1$ for each such component. For this reason, the contribution of each component to k_{2m+3} is equal to -2 times the product of the degree sums. Consequently, taking the sum over all components, we have the proof. □

Example 3.7. *Let us apply Theorem 3.6 to the graph* G *given in Fig. 3.*

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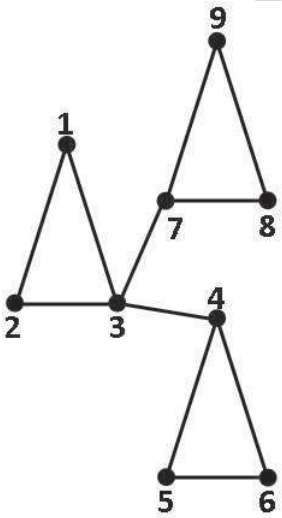


Figure 3: Graph G

The sum-edge characteristic polynomial of the graph G is

$$x^9 - 318x^7 - 688x^6 + 25576x^5 + 119216x^4 - 460192x^3 - 4204928x^2 - 9467392x - 6502400.$$

If we equalize this equation to the

$$x^9 + k_1x^8 + k_2x^7 + k_3x^6 + k_4x^5 + k_5x^4 + k_6x^3 + k_7x^2 + k_8x + k_9,$$

then by Theorem 3.6, we see that $k_1 = 0$, and we will look for k_3.

For k_3, all elementary subgraphs of G with three vertices are $E(G'_1) = \{123\} = \{12, 23, 31\}$, $E(G'_2) = \{456\} = \{45, 56, 64\}$ and $E(G'_3) = \{789\} = \{78, 89, 97\}$. Hence, $E(G'_1) = \{123\} = \{12, 23, 31\}$ contributes to k_3 by $(-1)^1 2^1 4.6.6 = -288$, $E(G'_2) = \{456\} = \{45, 56, 64\}$ contributes to k_3 by $(-1)^1 2^1 5.4.5 = -200$, and $E(G'_3) = \{789\} = \{78, 89, 97\}$ contributes to k_3 $(-1)^1 2^1 5.4.5 = -200$. Thus, the sum of the three, namely k_3 is equal to -688 and

$$\frac{-k_3}{2\prod_{\substack{i,j=1\\ j>i}}^{9}[d(v_i) + d(v_j)]} \neq 3$$

where v_i and v_j are vertices which are adjacent to each other in the component of one of the elementary subgraphs with 3 vertices. Moreover, you can observe that the value of the subgraphs G'_1, G'_2 and G'_3 are not the same.

The previous theorem is not verified because the graph G is not regular.

Example 3.8. *Let us apply the Theorem 3.6 to the complete graph K_4 given in Fig. 4:*

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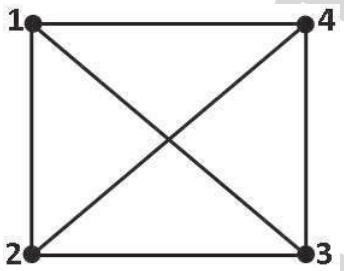


Figure 4: Complete Graph K_4

The sum-edge characteristic polynomial of the graph K_4 is

$$x^4 - 216x^2 - 1728x - 3888.$$

If we equalize this equation to the

$$x^4 + k_1x^3 + k_2x^2 + k_3x + k_4,$$

then by Theorem 3.6, we see that $k_1 = 0$ and now we can look for k_3.

For k_3, all elementary subgraphs of G with three vertices are $E(G'_1) = \{123\} = \{12, 23, 31\}$, $E(G'_2) = \{124\} = \{12, 24, 41\}$, $E(G'_3) = \{234\} = \{23, 34, 42\}$ and $E(G'_4) = \{134\} = \{13, 34, 41\}$. Hence each of $E(G'_1) = \{123\} = \{12, 23, 31\}$, $E(G'_2) = \{124\} = \{12, 24, 41\}$, $E(G'_3) = \{234\} = \{23, 34, 42\}$ and $E(G'_4) = \{134\} = \{13, 34, 41\}$ contributes to k_3 by $(-1)^1 2^1 6.6.6 = -432$.

As a result, $k_3 = -1728$ and

$$\frac{-k_3}{2\prod_{\substack{i,j=1\\ j>i}}^{4}[d(v_i) + d(v_j)]} = \frac{-(-1728)}{2.6.6.6} = 4$$

which is the number of $3-$cycles where v_i and v_j are vertices which are adjacent to each other in the component of one of the elementary subgraphs with 3 vertices. Consequently, Theorem 3.6 is verified because K_4 is regular.

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