Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.
https://doi.org/10.1007/s40010-022-00778-9

SCIENTIFIC RESEARCH PAPER

On α-Split Domination in Graphs

S. Amutha[1] · K. Suriya Prabha[2] · N. Anbazhagan[2] · S. Sankara Gomathi[3] · Ismail Naci Cangul[4]

Received: 15 January 2021 / Revised: 11 March 2022 / Accepted: 14 March 2022
© The Author(s), under exclusive licence to The National Academy of Sciences, India 2022

Abstract In this paper, we present the concept of α-split domination in graphs. Besides, we calculate the α-split domination number of path, cycle, complete bipartite graphs and discuss the upper bounds for α-split domination number in terms of order p, size q and the maximum degree Δ. We prove that α-split domination number is equal to the vertex covering number with $1 \geq \alpha > (\Delta(G) - 1)/\Delta(G)$ and propose the upper bound for α-split domination number of a tree with $\text{diam}(G) \geq 3$ in terms of the leaves among all its vertices and equality holds if and only if the split domination number of a tree is equal to the leaves among all its vertices. Also, we probe the α-split domination vertex critical graph with set of all support vertices of G and the lower bound for α-split domination number in terms of the order p and α satisfies $0 < \alpha \leq 1$. Moreover, we investigate the α-split domination edge criticalness and show that G is 2-$\gamma_{\alpha s}^{-}$-edge critical with $\delta(G) > 1$ if and only if $G = C_3$.

Keywords Domination · Domination number · Split domination · α-domination

Mathematics Subject Classification 05C38 · 05C40 · 05C69

✉ Ismail Naci Cangul
cangul@uludag.edu.tr

S. Amutha
amuthas@alagappauniversity.ac.in

K. Suriya Prabha
suriyaprabha23@gmail.com

N. Anbazhagan
anbazhagann@alagappauniversity.ac.in

S. Sankara Gomathi
ssgomathi@gmail.com

[1] Ramanujan Centre for Higher Mathematics, Alagappa University, Karaikudi 630 003, Tamilnadu, India

[2] Department of Mathematics, Alagappa University, Karaikudi, Tamilnadu 630 003, India

[3] Department of ECE, Adhi College of Engineering and Technology, Chennai, Tamilnadu, India

[4] Department of Mathematics, Faculty of Arts and Science, Bursa Uludag University, Gorukle Campus, 16059 Bursa, Turkey

1 Significance of the Work

Domination is one of the important concepts in graph theory due to its applications. In this paper, α-split domination is introduced. This domination is studied for several main graph classes and some upper bounds for the α-split domination number are obtained. Also several topics related to this domination type are studied.

2 Introduction

Graph terminology used in this paper can be found in [1]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order $p = |V|$ and edge set $E = E(G)$ of size $q = |E|$ having no isolated vertex. Let v be a vertex in V. The degree of v denoted by $d(v)$ is the number of edges of G incident to v. The minimum and maximum degree among the vertices of G are respectively denoted by $\delta(G)$ and $\Delta(G)$. For a subset $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. A path on p vertices is denoted by P_p, while a cycle on p vertices is denoted by C_p.

Published online: 16 April 2022

The fundamentals of domination were introduced in [2]. The open neighborhood of v is $N_G(v) = \{u \in V | uv \in E\}$ and the closed neighborhood of v is $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V$, its open neighborhood is the set $N_G(S) = \cup_{v \in S} N_G(v)$, and its closed neighborhood is the set $N_G[S] = N_G(S) \cup S$.

The concept of split domination in graphs can be found in [3]. In [4], split domination number of a congruent dominating graph is introduced. The notion of split domination number of a divisible dominating graphs was discussed by Amutha et al. in [5].

Prabha et al. [6] gave the idea of the neighborhood outer split domination in graphs. The α-domination in graphs has been scrutinized in [7]. Some remarks on α-domination is given in [8]. In [9, 10], the concepts of vertex removal and edge removal in α-domination were used. The α-domination and the α-total domination in graphs were introduced in [11, 12]. Motivated by this work, in this paper, we present the concept of α-split domination in graphs. For some α with $0 < \alpha \leq 1$, a split dominating set S in G is an α-split dominating set if for every vertex $v \in V \backslash S$, $|N(v) \cap S| \geq \alpha |N(v)|$. The minimum cardinality of an α-split dominating set of G is called the α-split domination number of G and it is denoted by $\gamma_{\alpha s}(G)$. We calculate the α-split domination number of certain standard graphs and we acquire bounds for $\gamma_{\alpha s}(G)$. The set of all support vertices of G is denoted by $S(G)$. The vertex covering number of G is denoted by $\alpha_0(G)$ and the diameter of G is denoted by $diam(G)$.

3 Bounds on α-Split Domination Number

In this section, we compute the α-split domination number of path, cycle, complete bipartite graphs and give some bounds for the α-split domination number in terms of the order, size, degrees, vertex covering number and other graph parameters.

Observation For $0 < \alpha \leq 1/\Delta$, we have $\gamma_{\alpha s}(G) = \gamma_s(G)$.

Theorem 1 *For $p \geq 3$,*

$$\gamma_{\alpha s}(P_p) = \begin{cases} \left\lceil \frac{p}{3} \right\rceil & \text{if } 0 < \alpha \leq \frac{1}{2} \\ \left\lfloor \frac{p}{2} \right\rfloor & \text{if } \frac{1}{2} < \alpha \leq 1. \end{cases}$$

Proof Let $G = P_p$. If $0 < \alpha \leq \frac{1}{2}$, then by above observation, $\gamma_{\alpha s}(G) = \gamma_s(G) = \lceil \frac{p}{3} \rceil$. Hence, we assume that $\alpha > 1/2$, for otherwise the desired result follows. Let S be an arbitrary α-SDS in G and let $T = V \backslash S$. If u is an end vertex of the path, then $\lceil \alpha \cdot d(u) \rceil = 1$, while if u is an internal vertex of the path, then $\lceil \alpha \cdot d(u) \rceil = 2$. Thus, if $u \in V \backslash S$, then $N(u) \subseteq S$, implying that $|[S,T]| \leq (|T| - 3) + 2 = |T| - 1$, where $[S, T]$ denotes the set of all edges between S and T in G. Since each vertex in the SDS S is adjacent to at most two vertex in $V - S$, we have that $|[S,T]| \leq |S|$. Hence, $|S| \geq |T| - 1 = (p - |S|) - 1$ or equivalently, $|S| \geq \frac{p-1}{2}$. Since S is an arbitrary α-SDS, we have that $\gamma_{\alpha s}(G) \geq \lceil \frac{p-1}{2} \rceil$. To show that $\gamma_{\alpha s}(G) \leq \lceil \frac{p-1}{2} \rceil$, let G be the path $u_1, u_2, \cdots, u_p$ and let $S = \bigcup_{j=0}^{\lfloor \frac{p}{2} \rfloor - 1} \{u_{2j+2}\}$. If p is even, then $D = S$, where D is the arbitrary set of vertices in G and note that $|D| = \frac{p}{2}$. If p is odd, then let $D = S$ and note that $|D| = \lfloor \frac{p}{2} \rfloor$. In both cases, D is a α-SDS of G of cardinality $\lfloor \frac{p}{2} \rfloor$, implying that $\gamma_{\alpha s}(G) \leq \lfloor \frac{p}{2} \rfloor$. Consequently, $\gamma_{\alpha s}(G) = \lfloor \frac{p}{2} \rfloor$. □

Theorem 2 *For $p \geq 3$,*

$$\gamma_{\alpha s}(C_p) = \begin{cases} \left\lceil \frac{p}{3} \right\rceil & \text{if } 0 < \alpha \leq \frac{1}{2} \\ \left\lceil \frac{p}{2} \right\rceil & \text{if } \frac{1}{2} < \alpha \leq 1. \end{cases}$$

Proof Let $G = C_p$. If $0 < \alpha \leq \frac{1}{2}$, then by Observation 1, $\gamma_{\alpha s}(G) = \gamma_s(G) = \lceil \frac{p}{3} \rceil$. Hence, we assume that $\alpha > \frac{1}{2}$, for otherwise, the desired result follows. Let S be an arbitrary α-SDS in G and let $T = V \backslash S$. For every vertex $u \in V$, we have $\lceil \alpha \cdot d(u) \rceil = 2$. Thus, if $u \in V \backslash S$, then $N(u) \subseteq S$ implying that $|[S,T]| = |T|$, where $[S, T]$ denotes the set of all edges between S and T in G. Since each vertex in the SDS S is adjacent to at most two vertices in $V \backslash S$, we have that $|[S,T]| \leq |S|$. Hence, $|S| \geq |T| = (p - |S|)$ or equivalently, $|S| \geq \frac{p}{2}$. Since S is an arbitrary α-SDS, we have $\gamma_{\alpha s}(G) \geq \lceil \frac{p}{2} \rceil$. To show that $\gamma_{\alpha s}(G) \leq \lceil \frac{p}{2} \rceil$, let G be the cycle $u_1, u_2, \ldots, u_p, u_1$ and let $S = \bigcup_{j=0}^{\lfloor \frac{p}{2} \rfloor - 1} \{u_{2j+2}\}$. If p is even, then $D = S$, where D is the arbitrary set of vertices in G and note that $|D| = \frac{p}{2}$. If p is odd, then let $D = S$ and note that $|D| = \lceil \frac{p}{2} \rceil$. In both the cases, D is a α-SDS of G of cardinality $\lceil \frac{p}{2} \rceil$ implying that $\gamma_{\alpha s}(G) \leq \lceil \frac{p}{2} \rceil$. Consequently, $\gamma_{\alpha s}(G) = \lceil \frac{p}{2} \rceil$.

Theorem 3 *Let $K_{m,n}$ be the complete bipartite graph with $1 \leq m \leq n$ .Then for all α with $0 < \alpha \leq 1$, we have $\gamma_{\alpha s}(K_{m,n}) = min\{m, n\}$.*

Proof Let $G = K_{m,n}$ have partite sets A and B, where $|A| = m$ and $|B| = n$. Let $S = A$ be a $\gamma_{\alpha s}(G)$-set and let $S_A = S \cap A$ and $S_B = S \cap B$. Then, $|S_A| \geq 1$ and $|S_B| = 0$ and let $S = B$ be a $\gamma_{\alpha s}(G)$-set. Then, $|S_A| = 0$ and $|S_B| \geq 1$. Let $u \in V - S$. If $u \in A$, then $|S_B| = |N(u) \cap S| \geq |N(u)| = n$, while if $u \in B$, then analogously, $|S_A| = |N(u) \cap S| \geq |N(u)| = m$. If $A = S_A$, then $|S| \geq |A| = m$, while if $B = S_B$, then $|S| \geq |B| = n$. Thus, if $A = S_A$ or $B \neq S_B$, then $|S| \geq m$ and if $A \neq S_A$ or $B = S_B$, then $|S| \geq n$. Consequently, $\gamma_{\alpha s}(G) = |S| \geq min\{m, n\}$. To prove the reverse inequality, we observe that the set A is an α-SDS and so $\gamma_{\alpha s}(G) \leq m$ or if B is an α-SDS, then $\gamma_{\alpha s}(G) \leq n$. Consequently, $\gamma_{\alpha s}(K_{m,n}) = min\{m, n\}$. □

Theorem 4 *If G is a graph of order p* and size q *with no isolated vertex and with minimum and maximum degrees δ and Δ, respectively, then for $0<\alpha\le 1$, the following holds*:

(a) $\gamma_{\alpha s}(G)\ge \frac{\lceil \alpha\delta\rceil\, p}{\Delta+\lceil\alpha\delta\rceil}$

(b) $\gamma_{\alpha s}(G)\ge \frac{(\delta+\lceil\alpha\delta\rceil)\,p-2q}{\delta+\lceil\alpha\delta\rceil}$

(c) $\gamma_{\alpha s}(G)\ge \frac{2\alpha q}{(\alpha+1)\Delta}$

Proof Let S be a $\gamma_{\alpha s}(G)$-set and let $T=V\backslash S$. Let $G[S]$ and $G[T]$ denote the subgraphs of G induced by S and T, respectively. Recall that $[S, T]$ denotes the set of all edges between S and T in G.

(a) We count the number of edges $|[S, T]|$ between S and T in two ways. Counting the edges from S to T, we have $|[S,T]|\le \Delta|S|$, while counting the edges from T to S, we have

$$|[S,T]|\ge \sum_{u\in T}\lceil \alpha\, d(u)\rceil \ge \lceil\alpha\delta\rceil\cdot|T| = \lceil\alpha\delta\rceil(p-|S|).$$

Thus, $\Delta\cdot|S|\ge\lceil\alpha\delta\rceil(p-|S|)$ or equivalently $\gamma_{\alpha s}(G)=|S|\ge(\lceil\alpha\delta\rceil p)/(\Delta+\lceil\alpha\delta\rceil)$. This proves (a).

(b) Since $G[S]$ contains an isolated vertex, we have $q(G[S])+|S|\ge 1$. Moreover $\sum_{u\in T} d_G(u)=2q(G[T])+|[S,T]|$ and $|T|=p-|S|$. As shown in (a) above, counting the edges from T to S, we have $|[S,T]|\ge\lceil\alpha\delta\rceil(p-|S|)$. Hence

$$\begin{aligned} q &= q(G[S])+|S|+q(G[T])+|[S,T]|-|S| \\ &\ge 1+\frac{1}{2}\left(\sum_{u\in T} d_G(u)-|[S,T]|\right)+|[S,T]|-|S| \\ &\ge 1+\frac{1}{2}\left(\sum_{u\in T}\delta\right)+\frac{1}{2}|[S,T]|-|S| \\ &\ge 1+\frac{1}{2}\delta(p-|S|)+\frac{1}{2}|[S,T]|-|S| \\ &\ge 1+\frac{1}{2}(1-\delta)|S|+\frac{1}{2}\lceil\alpha\delta\rceil(p-|S|)-|S| \end{aligned}$$

and so $2q-(\delta+\lceil\alpha\delta\rceil)p\ge(-\delta-\lceil\alpha\delta\rceil)|S|$. Since $-\delta-\lceil\alpha\delta\rceil<0$, we have $\gamma_{\alpha s}(G)=|S|\ge((\delta+\lceil\alpha\delta\rceil)p-2q)/(\delta+\lceil\alpha\delta\rceil)$. This proves (b).

(c) Counting the edges from S to T, we have $|[S,T]|\le\sum_{u\in S} d_G(u)$ since $G[S]$ has an isolated vertex. While counting the edges from T to S, we have $|[S,T]|\ge\sum_{u\in T}\alpha d(u)$. Hence, $\sum_{u\in S} d(u)\ge\sum_{u\in T}\alpha d(u)$. Adding $\sum_{u\in S}\alpha d(u)$ to both sides of inequality, we have $((\alpha+1)\Delta)|S|\ge\sum_{u\in S}(\alpha+1)d(u)\ge\sum_{u\in V}\alpha d(u)=2\alpha q$, and so $\gamma_{\alpha s}(G)=|S|\ge(2\alpha q)/((\alpha+1)\Delta)$. This proves (c).

□

Theorem 5 *If G is a graph of order p, size q having no isolated vertex and minimum degree $\delta\ge 2$ and if α satisfies $0<\alpha\le 1-1/\delta$, then $\gamma_{\alpha s}(G)\le\frac{2q-\lceil\alpha\delta\rceil\,p}{\delta-\lceil\alpha\delta\rceil}$.*

Proof Let S be a $\gamma_{\alpha s}(G)$-set and let $T=V\backslash S$. Let $G[S]$ and $G[T]$ denote the subgraphs of G induced by S and T, respectively. Since

$$\begin{aligned} q &\ge q(G[S])+|[S,T]| = \frac{1}{2}\left(\sum_{u\in S} d_G(u)-|[S,T]|\right)+|[S,T]| \\ &\ge \frac{1}{2}\delta|S|+\frac{1}{2}|[S,T]|\ge\frac{1}{2}\delta|S|+\frac{1}{2}\lceil\alpha\delta\rceil(p-|S|), \end{aligned}$$

we have $2q-\lceil\alpha\delta\rceil p\ge(\delta-\lceil\alpha\delta\rceil)|S|$. By assumption, $\alpha\le(\delta-1)/\delta$, and so $\delta-\lceil\alpha\delta\rceil>0$ implying that $\gamma_{\alpha s}(G)=|S|\le(2q-\lceil\alpha\delta\rceil p)/(\delta-\lceil\alpha\delta\rceil)$.

Theorem 6 *Let G be a graph. If $1\ge\alpha>(\Delta(G)-1)/\Delta(G)$, then $\gamma_{\alpha s}(G)=\alpha_0(G)$.*

Proof Suppose G is a graph and S is a $\gamma_{\alpha s}$-set, with $\alpha>(\Delta(G)-1)/\Delta(G)$. Let u be a vertex with $u\notin S$. Then, $|N(u)\cap S|>(\Delta(G)-1)|N(u)|/\Delta(G)$. Thus, we deduce that $\Delta(G)|N(u)\cap S|>\Delta(G)|N(u)|-|N(u)|$. Let $i\ge 0$ be an integer satisfying $i=|N(u)|-S$. Then, $|N(u)|-i=|N(u)\cap S|$ and so $\Delta(G)[|N(u)|-i]>\Delta(G)|N(u)|-|N(u)|$. Thus, $\Delta(G)\cdot i<|N(u)|$. If $i\ge 1$, this contradicts the definition of maximum degree. Hence, $i=0$. Since u is chosen to be an arbitrary vertex outside S, it must be true that every vertex in $V\backslash S$ has its open neighborhood contained in S. So there is no edge in G with both end vertices in $V\backslash S$. Consequently, S is a vertex cover for G and $\alpha_0(G)\le|S|$. Since the reverse inequality always holds, we must have equality. □

Proposition 1 *Let T be a tree on p vertices with l leaves and with $diam(T)\ge 3$. Also let α satisfy $0<\alpha\le 1$. Then, $\gamma_{\alpha s}(T)\le p-l$ with equality if and only if $\gamma_s(T)=p-l$.*

Proof Let $T=(V,E)$. For every leaf u of T, we have $\lceil\alpha\cdot d(u)\rceil=1$. Since $diam(T)\ge 3$, every vertex that is not a leaf has at least one non-leaf neighbor. Hence, the set $V\backslash L(T)$ is an α-SDS, and so $\gamma_{\alpha s}(T)\le p-l$. By Observation 1, we have that $\gamma_s(T)\le\gamma_{\alpha s}(T)$. Hence, if $\gamma_s(T)=p-l$, then $\gamma_{\alpha s}(T)=p-l$. Conversely, suppose that $\gamma_s(T)<p-l$. Let D be a $\gamma_s(T)$-set. If D contains a leaf v, then we can simply replace v by a non-leaf neighbor of the support vertex adjacent to u to obtain a new $\gamma_s(T)$-set that contains fewer leaves. Hence, we may choose the $\gamma_s(T)$-set to contain no leaf. Every SDS in a graph contains the support vertices but no leaf, there must be a vertex $u\notin D$ that is neither a leaf nor a support vertex. Let $S=V\backslash(L(T)\cup\{u\})$. Since $D\subseteq S$, the set S is an SDS in T. Further, since $N(u)\subseteq S$ and since S contains a support vertices, every vertex outside S has all its neighbors in S

implying that the set S is an α-SDS and $\gamma_{\alpha s}(T) \leq |S| < p - l$. Hence, if $\gamma_s(T) < p - l$, then $\gamma_{\alpha s}(T) < p - l$. Therefore, if $\gamma_{\alpha s}(T) = p - l$, then $\gamma_s(T) = p - l$.

We call a graph G as α-split domination vertex critical or just $\gamma_{\alpha s}$-vertex critical if for any $v \in V(G) - S(G)$, $\gamma_{\alpha s}(G - v) < \gamma_{\alpha s}(G)$, where $S(G)$ is the set of all support vertices of G.

Theorem 7 *If G is a $\gamma_{\alpha s}$-vertex critical graph of order p, then for any vertex $u \in V(G) - S(G)$,*

$$\gamma_{\alpha s}(G) \geq \left\lceil \frac{\alpha\, \delta(G-u)\, p \;+\; \Delta(G)}{\alpha\, \delta(G-u) \;+\; \Delta(G} \right\rceil.$$

Proof Let G be a $\gamma_{\alpha s}$-vertex critical graph of order p and let $u \in V(G) - S(G)$. Let $H = G - u$ and let D be a $\gamma_{\alpha s}(H)$-set. Then $|D| = \gamma_{\alpha s}(G) - 1$. Let T be the set of edges between D and $V(H) - D$. By counting the edges from D to $V(H) - D$, we obtain that

$$|T| \leq \sum_{u \in D} \deg(u) \leq |D|\Delta(G).$$

On the other hand, since D is an α-split dominating set for H, we find that

$$|T| \geq \sum_{u \in V(H)-D} \alpha \deg_H(u) \geq \alpha\delta(H)(|V(H)| - |D|).$$

Now, we obtain $|D|\ \Delta(G) \geq \alpha\delta(H)(p - 1 - |D|)$. Since $|D| = \gamma_{\alpha s}(G) - 1$ and $H = G - u$, a simple calculation implies that

$$\gamma_{\alpha s}(G) \geq \left\lceil \frac{\alpha\delta(G-u)p + \Delta(G)}{\alpha\delta(G-u) + \Delta(G} \right\rceil.$$

□

Theorem 8 *If G is a $\gamma_{\alpha s}$-vertex critical graph of order p and size q, then*

$$\gamma_{\alpha s}(G) \geq \left\lceil \frac{2\alpha q - \alpha\, \Delta(G) \;+\; \Delta(G)}{\Delta(G)\ (\alpha + 1)} \right\rceil.$$

Proof Let G be a $\gamma_{\alpha s}$-vertex critical graph of order p and size q. Let $u \in V(G) - S(G)$ and $H = G - u$. Let D be a $\gamma_{\alpha s}(H)$-set. Then,

$$\sum_{u \in D} \deg_H(u) \geq \sum_{u \in V(H)-D} \alpha \deg_H(u).$$

Now,

$$\begin{aligned}(\alpha+1)|D|\Delta(G) &\geq \alpha \sum_{u \in D} \deg_H(u) + \sum_{u \in D} \deg_H(u) \\ &\geq \alpha \sum_{u \in D} \deg_H(u) + \sum_{u \in V(H)-D} \alpha \deg_H(u) \geq \alpha \sum_{u \in V(H)} \deg_H(u) \\ &= \alpha(2q - 2\deg_G(u)) \geq \alpha(2q - 2\Delta(G)).\end{aligned}$$

Since $|D| = \gamma_{\alpha s}(G) - 1$, a simple calculation completes the proof.

We call a graph G, $\gamma^-_{\alpha s}$-edge critical if $\delta(G) > 1$ and removing any edge of G decreses the α-split domination number. □

Theorem 9 *If a graph G is $\gamma^-_{\alpha s}$-edge critical, then for any edge $e = uv$ and any $\gamma_{\alpha s}(G - e)$-set D, the following hold.*

(i) $\{u, v\} \cap D = \phi$
(ii) $|N_G(v) \cap D| < \alpha \deg_G(v)$
. or $|N_G(u) \cap D| < \alpha \deg_G(u)$.

Proof

(i) Let G be a $\gamma^-_{\alpha s}$-edge critical graph. Let $e = uv$ be an edge of G and let D be a $\gamma_{\alpha s}(G - e)$-set. If $\{u, v\} \cap D \neq \phi$, then D is an α-split dominating set for G, a contradiction. So, $\{u, v\} \cap D = \phi$.
(ii) If $|N_G(v) \cap D| \geq \alpha \deg_G(v)$ and $|N_G(u) \cap D| \geq \alpha \deg_G(u)$, then D is an α-split dominating set for G, a contradiction. Thus, $|N_G(v) \cap D| < \alpha \deg_G(v)$ or $|N_G(u) \cap D| < \alpha \deg_G(u)$. This completes the proof.

For an integer $k \geq 2$, a graph G is called $k - \gamma^-_{\alpha s}$ - edge critical if G is $\gamma^-_{\alpha s}$ - edge critical with $\gamma_{\alpha s}(G) = k$. □

Theorem 10 *If G is a graph with $\delta(G) > 1$, then G is 2-$\gamma^-_{\alpha s}$-edge critical if and only if $G = C_3$ and $0 < \alpha \leq 1$.*

Proof It is easy to see that $G = C_3$ is 2 - $\gamma^-_{\alpha s}$-edge critical for $0 < \alpha \leq 1$. Assume that G is a 2 - $\gamma^-_{\alpha s}$-edge critical graph of order p with $\delta(G) > 1$. Let $e = xy \in E(G)$. Then, $\gamma_{\alpha s}(G - e) = 1$. Let $D = \{a\}$ be a $\gamma_{\alpha s}(G - e)$-set. Clearly, $a \notin \{x, y\}$. Then, $\deg(a) = p - 1$. We observe that $1 = |N(y) \cap D| \geq \alpha(\deg(y) - 1)$, and so $\deg(y) \leq \frac{1}{\alpha} + 1$ and similarity $deq(x) \leq \frac{1}{\alpha} + 1$. Furthermore, if $b \in V(G) - \{a, x, y\}$, then

$$\deg(b) \leq \frac{1}{\alpha}. \tag{1}$$

We consider the following two cases.

(i) Assume that $\deg(x) = \deg(y)$. If $p = 3$, then $G = C_3$, and it is $\gamma^-_{\alpha s}$-edge critical if and only if $0 < \alpha \leq 1$. If $p \geq 4$, let $z \in V(G) - \{a, x, y\}$. Since $\gamma_{\alpha s}(G - az) = 1$, we find that $\deg(x) \leq \frac{1}{\alpha}$ or $\deg(y) \leq \frac{1}{\alpha}$. Since $\deg(x) = \deg(y)$, we obtain $\deg(x) = \deg(y) \leq \frac{1}{\alpha}$. But then, $\{a\}$ is an α split dominating set for G, implying the contradiction $\gamma_{\alpha s}(G) = 1$.
(ii) Assume that $\deg(x) \neq \deg(y)$. Assume, without loss of generality that $\deg(x) > \deg(y)$. Let $x_1 \in N(x) - \{a, y\}$, and let D_1 be a $\gamma_{\alpha s}(G - ax_1)$-set. If $D_1 \neq \{x\}$, then $\deg(x) \leq \frac{1}{\alpha}$ and so $\deg(y) < \deg(x) \leq \frac{1}{\alpha}$. Consequently, $D = \{a\}$ is an α-split dominating set for G, implying that $\gamma_{\alpha s}(G)$=1, a contradiction. Thus, $D_1 = \{x\}$. So $\deg(x) = p - 1$. Let D_2 be a $\gamma_{\alpha s}(G - ax)$-set. Then, $D_2 = \{u\}$ for some $u \in$

$V(G) - \{a, x\}$ with $\deg(u) = p - 1$. Since $\deg(y) < p - 1$, we find that $u \neq y$. But by 1, $\deg(u) \leq \frac{1}{\alpha}$. Thus,

$$\deg(x) = p - 1 = \deg(u) \leq \frac{1}{\alpha}$$

and

$$\deg(y) \leq \frac{1}{\alpha}.$$

Now, $\{a\}$ is an α-split dominating set for G, implying that $\gamma_{\alpha s}(G) = 1$, a contradiction.

□

4 Conclusion

In this paper, the exact value for the α-split domination number of P_p, C_p, $K_{m,n}$ with $0 < \alpha \leq 1$ is calculated. Also, we proved that α-split domination number is equal to the vertex covering number with $1 \geq \alpha > (\Delta(G) - 1)/\Delta(G)$. Moreover, we bestowed the bounds for α-split domination of vertex critical graphs with set of all support vertices of G and edge critical graphs. In future, we plan to establish upper and lower bounds for the various domination parameters of a graph.

References

1. Harary F (1972) Graph theory. Addison Wesley, Reading Mass
2. Haynes TW, Hedetniemi ST, Slater PJ (1998) Fundamentals of domination in graphs. Marcel Dekker Inc., New York
3. Chelvam TT, Chellathurai SR (2009) A note on split domination number of a graph. J Discrete Math Sci Cryptogr 12(2):179–186
4. Prabha KS, Amutha S (2018) Split domination number of a congruent dominating graphs. Int J Pure Appl Math 119(12):14633–14642
5. Amutha S, Prabha KS, Anbazhagan N, Shanthi P (2020) Split domination number of a divisible dominating graphs. J Discrete Math Sci Cryptogr. https://doi.org/10.1080/09720529.2020.1744813
6. Prabha KS, Amutha S, Anbazhagan N, Gomathi SS (2018) Neighborhood outer split domination in graphs. J Discrete Math Sci Cryptogr 22(5):787–799
7. Das A, Laskar RC, Rad NJ (2018) On α-domination in graphs. Graphs Comb 34(1):193–205
8. Dahme F, Rautenbach D, Volkmann L (2004) Some remarks on α-domination. Discuss Math Graph Theory 24(3):423–430
9. Rad NJ, Volkmann L (2016) Edge-removal and edge-addition in α domination. Graphs Comb 32(3):1155–1166
10. Rad NJ, Volkmann L (2012) Vertex removal in α domination. Filomat 26(6):1257–1262
11. Henning MA, Rad NJ (2012) On α-total domination in graphs. Discrete Appl Math 160(7–8):1143–1151
12. Dunbar JE, Hoffman DG, Laskar RC, Markus LR (2000) α domination. Discrete Math 211(1–3):11–26

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.