Some Properties of the Minimal Polynomials of $2cos(\pi/q)$ for odd q

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Abstract. The number $\lambda_q = 2\cos\pi/q$, $q \in \mathbf{N}$, $q \geq 3$, appears in the study of Hecke groups which are Fuchsian groups, and in the study of regular polyhedra. There are many results about the minimal polynomial of this algebraic number. Here we obtain the minimal polynomial of this number by means of the better known Chebycheff polynomials for odd q and give some of their properties.

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INTRODUCTION

For $n \in \mathbf{N}$, the n-th Chebycheff polynomial $T_n(x)$ is defined by

$$T_n(x) = \cos(n\cos^{-1}x),$$

where $x \in \mathbf{R}$, $|x| \leq 1$, or $T_n(\cos\theta) = \cos(n\theta)$, $\theta \in \mathbf{R}$. We get a normalisation of T_n given by $D_n(x) = 2T_n(x/2) = 2\cos(n\arccos(x/2)) = 2\cos n\theta$ where $x = D_1(x) = 2\cos\theta; x, \theta \in \mathbf{R}$, $|x| \leq 2$, $n \in \mathbf{N}$.

The minimal polynomial of $\lambda_q = 2\cos(\pi/q)$ for odd q is given in [1, 3]. In this work we search for the minimal polynomial of this algebraic number λ_q when q is even and find this polynomial in terms of normalized Chebycheff polynomials. The number λ_q plays an important role in the theory of Hecke groups, which are the discrete subgroups of $PSL(2,\mathbf{R})$ generated by two linear fractional transformations

$$R(z) = -1/z \qquad and \qquad T(z) = z + \lambda_q,$$

see [1]. λ_q is also used in the geometry of polyhedra.

The polynomials D_n are explicitly given in [4] as

$$D_n(x) = \sum_{i=0}^{[n/2]} (-1)^i \frac{n}{n-i} \left(\begin{array}{c} n-i \\ i \end{array} \right) x^{n-2i}$$

where $[\alpha]$ denotes the greatest integer less than or equal to α. Therefore $\deg(D_n(x)) = n$.

The first few D_n's are $D_1(x) = x$, $D_2(x) = x^2 - 2$, $D_3(x) = x^3 - 3x$, $D_4(x) = x^4 - 4x^2 + 2$, $D_5(x) = x^5 - 5x^3 + 5x$. For convenience we define $D_0(x) = 2$.

In [1], the fourth author obtained the minimal polynomial P_q^* of λ_q by means of the T_n's and D_n's. It was also proven that the degree of $P_q^*(x)$ is $\varphi(2q)/2$, where $\varphi(q)$ denotes the Euler function. Here we give a relatively simpler formula for P_q^*.

Let $\zeta = e^{i\pi/q}$ so that $\lambda_q = \zeta + \zeta^{-1}$. Then $\zeta^q + 1 = 0$. As q is odd, $(\zeta+1)(\zeta^{q-1} - \zeta^{q-2} + ... - \zeta + 1) = 0$, i. e. $\zeta^{q-1} - \zeta^{q-2} + ... - \zeta + 1 = 0$. Hence

$$\zeta^{(q-1)/2}(\zeta^{(q-1)/2} - \zeta^{(q-3)/2} + ... - \zeta^{(3-q)/2} + \zeta^{(1-q)/2}) = 0,$$

i.e.

$$(\zeta^{(q-1)/2} + \zeta^{(1-q)/2}) - (\zeta^{(q-3)/2} + \zeta^{(3-q)/2}) + ... - \varepsilon(\zeta + \zeta^{-1}) + \varepsilon = 0$$

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where

$$\varepsilon = \begin{cases} +1 & if\ q \equiv 1 \mod 4 \\ -1 & if\ q \equiv -1 \mod 4 \end{cases}.$$

The last equation is equivalent to $\sum_{k=0}^{(q-1)/2}(-1)^k D_{\frac{q-1}{2}-k} = 0$. From now on we denote the polynomial on the left hand side by $P_q(x)$. We have $\deg P_q(x) = \frac{q-1}{2}$.

We denote by W_q (W_q^* respectively) the set of roots of P_q (P_q^* respectively) in the simple extension $\mathbf{Q}(\lambda_q)$. Then

Theorem 1 $W_q = \{2\cos\pi/q,\ 2\cos 3\pi/q, ..., 2\cos(q-2)\pi/q\}$.

Proof $\zeta^q = -1 = e^{i\pi}$, so $\zeta_k = e^{i(\pi/q+2k\pi/q)}, k = 1,2,...,q-1$. But $e^{i\pi/q}.e^{i(2q-1)\pi/q} = 1$, $e^{i3\pi/q}.e^{i(2q-3)\pi/q} = 1, ..., e^{i(q-2)\pi/q}.e^{i(q+2)\pi/q} = 1$ and $e^{i\pi/q} + e^{-i\pi/q} = \lambda_q$, $e^{i3\pi/q} + e^{-i3\pi/q} = 2\cos 3\pi/q, ..., e^{i(q-2)\pi/q} + e^{i(q+2)\pi/q} = 2\cos(q-2)\pi/q$. Therefore

$$P_q(x) = (x - 2\cos\pi/q)(x - 2\cos 3\pi/q)...(x - 2\cos(q-2)\pi/q).$$

Theorem 2 *If $p \mid q$ then $W_p \subset W_q$.*

Proof Let $q = p.k$, $k \in \mathbf{N}$. Then

$$W_p = \{2\cos k\pi/q,\ 2\cos 3k\pi/q, ..., 2\cos(q-2)k\pi/q\}$$

and

$$W_q = \{2\cos\pi/q,\ 2\cos 3\pi/q, ..., 2\cos(q-2)\pi/q\}.$$

We want to show that the first and the last elements in W_p are in W_q implying the result as the coefficients of the angles π/q and $k\pi/q$ are successive odd numbers. As k is odd and $k \mid q$, $2\cos k\pi/q \in W_q$. Consider $2\cos(p-2)k\pi/q$. As $(p-2)k < p.k = q$ and $(p-2)k$ is odd, this is also in W_q.

We now built the minimal polynomial P_q^* successively, beginning with prime q.

Theorem 3 *Let $q = p$ be an odd prime. Then P_p is irreducible and $P_p^* = P_p$.*

Proof By (1), $\deg P_p(x) = \frac{p-1}{2}$. Also $\varphi(2p)/2 = \frac{p-1}{2}$. Now P_q and P_q^* are both polynomials of λ_q, the latter being minimal. In other words we obtain P_q^* by dividing P_q by several factors. As they both have the same degree, they must be equal.

Theorem 4 *Let $q = p_1.p_2$ with p_1, p_2 distinct odd primes. Then P_q is reducible and*

$$P_q^* = \frac{P_q}{P_{p_1}^*.P_{p_2}^*} = \frac{P_q}{P_{p_1}.P_{p_2}}.$$

Proof By Theorem 2, $W_{p_i} \subset W_q$ for $i = 1,2$, i.e. both P_{p_1} and P_{p_2} divide P_q. As polynomials form an Eulerian ring we have

Lemma 5 $(p_1, p_2) = 1$ *iff* $W_{p_1} \cap W_{p_2} = \emptyset$.

By Lemma 5 we guarantee that W_{p_1} and W_{p_2} have no common element implying that no linear factor divides P_q twice when it is divided by P_{p_1} and P_{p_2} together. This is necessary, and in fact very important, as all roots of P_q are distinct.

Let P'_q denote the polynomial obtained from P_q by dividing it by P_{p_1} and P_{p_2}. It has degree

$$\begin{aligned}\deg P'_q(x) &= \frac{q-1}{2} - \left(\frac{p_1-1}{2} + \frac{p_2-1}{2}\right) \\ &= \frac{(p_1-1)(p_2-1)}{2}\end{aligned}$$

which is equal to the $\deg P^*_q(x)$ of the minimal polynomial P^*_q. Hence the result follows. Let us see this with an example.

Theorem 6 *Let $q = p^2$, p odd prime. Then P_q is reducible and $P^*_q = \frac{P_q}{P_p}$.*

Proof It is similar to the proof of Theorem 3.

Note that Theorems 4 and 1 suggest us to divide P_q by P_d for each divisor d, to obtain the minimal polynomial P^*_q. This process is necessary but not always enough as we shall see. In these theorems, $q = p_1.p_2$ and $q = p^2$ and therefore all proper divisors of q are prime. In fact these two are the only values of q having this property. For all other q, there are some non-prime divisors of q. Some of those will not be comprime. If d_1 and d_2 are two such divisors with $(d_1, d_2) = d > 1$, then P_{d_1} and P_{d_2} are not comprime either. Similarly to Lemma 5, we have

Lemma 7 *Let q be odd. Let d_1, $d_2 \mid q$ with $(d_1, d_2) = d > 1$. Then*

$$W_{d_1} \cap W_{d_2} \neq \emptyset.$$

In fact

Lemma 8 *Let q be odd. Let d_1, $d_2 \mid q$. Then $(d_1, d_2) = d > 1$ iff*

$$W_{d_1} \cap W_{d_2} = W_d \neq \emptyset.$$

Proof As $d \mid d_i$ for $i = 1, 2$, by Theorem 2, $W_d \subset W_{d_i}$ and therefore

$$W_d \subset W_{d_1} \cap W_{d_2}.$$

Let now $\alpha \in W_{d_1} \cap W_{d_2}$. Then $\alpha \in W_{d_1}$ and $\alpha \in W_{d_2}$ and hence α is a root of P_{d_1} and P_{d_2}. Let $\alpha = 2\cos m\pi/n$, such that m, n are comprime. Here $m < n$, necessarily. As α is a root of P_{d_1} and P_{d_2}, $n \mid d_1$ and $n \mid d_2$. Therefore $n \mid (d_1, d_2) = d$. Let $d = l.n$, $l \in \mathbf{N}$. Then $\frac{1}{n} = \frac{l}{d}$ and therefore $\alpha = 2\cos lm\pi/d \in W_d$, as $l.m$ is odd and $1 < l.m \leq d-2$. Indeed $l.m < l.n = d$ and therefore $l.m \leq d-2$. Hence we have $W_d = W_{d_1} \cap W_{d_2}$. As $\lambda_d \in W_{d_1} \cap W_{d_2}$, $W_d \neq \emptyset$.

Recall that we divide P_q by P_{d_i}, $d_i \mid q$, to find P^*_q. When two such divisors are comprime then corresponding P polynomials have common factors. Therefore P_q is divided twice by the same factor. But we cannot do this as all factors of P_q are different. For this reason, to find the P^*_q, we must multiply P_q by P_d for every two comprime divisors d_i, d_j such that $(d_i, d_j) = d$, before dividing P_q by P_{d_i} and P_{d_j}. We can see the application of this in the following theorem:

Theorem 9 *Let $q = p^n$, p odd prime, $n \geq 2$, $n \in \mathbf{N}$. Then P_q is reducible and*

$$P^*_q = \frac{P_q}{P_{p^{n-1}}}.$$

Proof For $n = 2$ we have the result by Theorem 6. Now for $n = 3$, the only proper divisors of q are p and p^2. To obtain $P^*_{p^3}$, we first divide P_{p^3} by P_p and P_{p^2}. But as $W_p \subset W_{p^2}$, not to divide P_{p^3} twice by the same factors is only possible by dividing P_{p^2} by P_p first and then doing as before. *i.e.*

$$P^*_{p^3} = \frac{P_{p^3}}{P_p.\frac{P_{p^2}}{P_p}} = \frac{P_{p^3}}{P_{p^2}}$$

as required. Similarly

$$P^*_{p^n} = \frac{P_{p^n}}{P_p.\frac{P_{p^2}}{P_p}.\frac{P_{p^3}}{P_{p^2}}.\ldots.\frac{P_{p^{n-1}}}{P_{p^{n-2}}}} = \frac{P_{p^n}}{P_{p^{n-1}}}.$$

Theorem 10 *Let q be odd. Then*

$$P^*_q = \frac{P_q}{\prod\limits_{d|q,d\neq 1,q} P^*_d}.$$

Proof It follows from Lemma 8 and the discussion afterwards.

Note that the number of divisors P^*_d in the denominator could be made less using Theorems 1.4, 1.6 and 1.9. *e.g.* if $q = p_1^2.p_2$ then

$$P^*_q = \frac{P_q}{P^*_{p_1}.P^*_{p_2}.P^*_{p_1^2}.P^*_{p_1p_2}} = \frac{P_q}{P_{p_1}.P_{p_2}.\frac{P_{p_1^2}}{P_{p_1}}.\frac{P_{p_1p_2}}{P_{p_1}.P_{p_2}}} = \frac{P_q.P_{p_1}}{P_{p_1^2}.P_{p_1p_2}}.$$

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