Nursena Günhan Ay and Emrullah Yaşar*

On the bilinear Bäcklund transformation, Lax pair and interaction solutions to stochastic potential Korteweg–De Vries equation

https://doi.org/10.1515/jncds-2025-0105
Received September 19, 2025; accepted March 6, 2026; published online April 9, 2026

Abstract: In this study, the bilinear formalism, bilinear Bäcklund transformations (BBT) and Lax pair of the stochastic potential Korteweg–De Vries equation (spKdV) are established using the Bell polynomial approach. The spKdV equation, found in applications related to electric circuits and multi-component plasmas, is utilized to characterize the weak dispersive effects on the transmission of photons and nonlinear optical solitons. Using the bilinear neural network method (BNNM), we obtain the analytical solutions of the spKdV and discuss the physical behavior of these solutions with various graphical representations. In addition, we obtain the conservation vectors of the model using the multiplier method. We expect that the findings will yield a profound understanding of the fundamental dynamics and characteristics of the model.

Keywords: bilinear Bäcklund transformations; Lax pair; stochastic potential Korteweg–De Vries equation; bilinear neural network method

1 Introduction

Nature's inherent nonlinearity presents a significant challenge for understanding complex systems at their core. This perspective is widely endorsed within the research community. Investigating various forms of nonlinear partial differential equations (NLPDEs) is crucial for mathematically modeling dynamic systems that exhibit temporal variations.These models find applications across diverse fields, including engineering, climate science, computational biology, finance, optic, materials science, epidemiology, robotics, and atmospheric sciences. The exact or numerical solutions to these models, which explain phenomena across various fields, have become an important area of research, leading to the development of numerous methods by scientists working in the field.

The methods that have been developed include Lie symmetry analysis [1], Hirota bilinear method [2], the generalized exponential rational function approach [3], $\frac{G'}{G}$ expansion method [4], new generalized exponential rational function method [5], [6], F-expansion method [7], Adomian decomposition method [8], modified simple equation method [9] and Bell-polynomial approach [10], [11].

The Bell polynomial was developed by Bell in 1934 [10]. Subsequently, Lambert showd that [12], the Bell polynomial method can be used to obtain the Hirota bilinear form of NLPDEs, the BBT, the Lax pair, and conservation laws [13]. The Maple package program named PDEBellII, which constructs the bilinear forms, BBT, Lax pairs, and conservation laws of KdV-type soliton equations from the Bell polynomials scheme, was developed in 2014 [14].

In this study, we will analyze the spKdV, presented as a modified version of the potential KdV equation;

***Corresponding author: Emrullah Yaşar**, Department of Mathematics, Faculty of Arts and Sciences, Bursa Uludag University, 16059 Bursa, Turkiye, E-mail: eyasar@uludag.edu.tr
Nursena Günhan Ay, Department of Mathematics, Faculty of Arts and Sciences, Bursa Uludag University, 16059 Bursa, Turkiye; and Department of Mathematics, Faculty of Engineering and Natural Sciences, İstanbul Medeniyet University, 34700 Üsküdar, Istanbul, Turkiye, E-mail: nursena.gunhan@medeniyet.edu.tr

Open Access. © 2026 the author(s), published by De Gruyter. This work is licensed under the Creative Commons Attribution 4.0 International License.

$$u_t + \beta u_x^2 + \alpha u_x + \kappa u_{xxx} = 0 \tag{1}$$

where κ, α and β are named to as dispersion coefficient, stochastic parameter and nonlinearity coefficient, respectively. In Eq.(1), the case where $\alpha = 0$ corresponds to the potential-KdV equation, which is commonly used in water wave problems. Eq. (1), which is used in applications such as electrical circuits and multicomponent plasmas, is a model introduced to describe the weak dispersion effects in the transmission of photons ans nonlinear optical solitons [15], [16]. The integrability of the stochastic potential Korteweg–De Vries (spKdV) equation has recently attracted increasing attention and has been investigated through different analytical approaches, including the Lax pair formulation and infinite conservation laws. In this context, various complex-type solutions such as higher-order position, negaton, and hybrid solutions have been constructed via the Darboux transformation [17]. In addition, stochastic KdV-type models have been widely employed to describe nonlinear wave propagation in random or fluctuating media, where dispersive and stochastic effects play a significant role. In recent years, researchers have used several methods to study the stochastic potential KdV equation. For example, Lie symmetry analysis has helped derive similarity reductions and exact solutions for the spKdV equation. Conservation laws have also been built using Ibragimov's method [16]. Additionally, analysts have used the Hirota bilinear technique to find analytical solutions of the spKdV equation, resulting in various exact wave structures [15]. These studies show the complex mathematical nature of the spKdV equation and emphasize the usefulness of symmetry-based and bilinear methods for its analysis. In contrast to these existing studies, the present work is based on the bilinear formalism of the spKdV equation and uses the Bell polynomial approach to systematically construct its bilinear Bäcklund transformations along with the corresponding Lax pair. The integrability properties of the spKdV equation can be analyzed in a different and structurally transparent manner using this framework. Additionally, the bilinear neural network method (BNNM), which provides a versatile mechanism for managing the bilinear structure of stochastic nonlinear evolution equations, yields new classes of analytical solutions. The relevance of the suggested method in the study of stochastic nonlinear wave dynamics is demonstrated by the graphical analysis used to further examine the physical behavior of the resulting solutions and the multiplier method used to derive the corresponding conservation laws.

This work consist of four sections. The bilinear formalism for the spKdV is derived by the Bell polynomials approach in Section 2. At the same time we obtain, BBT and Lax pair for the spKdV. In the third section, the exact solutions of the model are obtained using BNNM [18] with 2-2-2-1 neural network architecture. In the next section we derive conservation laws of Eq. (1) using the multiplier approach. In the final section, we present various graphical representations to enhance the understanding of the physical structure of the obtained solutions, and we make some physical interpretations.

2 Bilinear formalism, BBT and Lax pair

In this section, firstly we will generate the bilinear form of Eq. (1) using the binary Bell polynomials technique. In the next step, we construct the BBT and the Lax pair of Eq. (1).

2.1 Bilinear formalism

Definition 2.1 [10], [14]. Assuming that $g = g(x_1, x_2, \dots, x_m)$ is a C^∞ function of multiple variables, multidimensional Bell polynomials are defined as follows:

$$\mathbb{Y}_{k_1,\dots,k_m}(g) \equiv \mathbb{Y}_{k_1,\dots,k_m}\left(\{g_{n_1x_1,\dots,n_mx_m} \mid 1 \le n_i \le k_i,\ 0 \le i \le m\}\right) = e^{-g}\frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_m}}{\partial x_m^{k_m}} e^{g},$$

where

$$g_{n_1x_1,\dots,n_mx_m} = \frac{\partial^{n_1}}{\partial x_1^{n_1}} \cdots \frac{\partial^{n_m}}{\partial x_m^{n_m}} g \quad \text{and} \quad g_{0x_i} \equiv g.$$

Theorem 1. *Using transformation*

$$u = 2(\ln(Q))_x \tag{2}$$

the $(1+1)$*-dimensional spKdV Eq.* (1) *can be bilinearized into*

$$\left[\kappa D_x^4 + \alpha D_x^2 + D_t D_x\right] Q.Q = 0. \tag{3}$$

Proof: Take into consideration the following formula to add a dimensionless field p:

$$u = c\,p_x \tag{4}$$

where c is free parameter to be determined. The aim is to choose c such that Eq. (1) is compatible with P-polynomials. Puting (4) into Eq. (1) we get;

$$E(p) = p_{xx}^2 \beta c + \kappa p_{xxxx} + \alpha p_{xx} + p_{xt} = 0. \tag{5}$$

Getting $c = \frac{3\kappa}{\beta}$, Eq. (5) may be expressed as a P-polynomial combination;

$$\alpha P_{0,2x}(p) + \kappa P_{0,4x}(p) + P_{x,t}(p) = 0. \tag{6}$$

Using the relationship between Hirota bilinear operators and P-polynomials given below;

$$\begin{aligned} Q^{-2} D_{x_1}^{i_1} \dots D_{x_m}^{i_m} Q.Q &= \mathbb{Y}_{i_1 x_1 \dots i_m x_m}(0, w = 2\ln(Q)) \\ &= \begin{cases} 0 & \sum_{j=1}^{m} n_j \text{ is odd} \\ P_{i_1 x_1 \dots i_m x_m}(w) & \sum_{j=1}^{m} n_j \text{ is even,} \end{cases} \end{aligned} \tag{7}$$

with a change of the dependent variable

$$w = p = 2\ln(Q) \iff u = p_x = 2(\ln(Q))_x \tag{8}$$

the bilinear formalism (3) for Eq. (1) may be constructed directly from equations (6)–(8). The bilinear operators are defined as;

$$D_x^i D_t^j A(x,t)B(x',t') = \left(\partial_x - \partial_{x'}\right)^i \left(\partial_t - \partial_{t'}\right)^j A(x,t)B(x',t')|_{x'=x,t'=t}. \tag{9}$$

2.2 BBT

In addition to finding solutions, BBT is also crucial for demonstrating the integrabability of a nonlinear evolution equation (NLEE). Below, BBT is obtained for Eq. (1) using binary Bell polynomials [10], [14].

Theorem 2. *If* Q *is a solution to the bilinear equation* (3)*, then* R*, which satisfies the following conditions:*

$$\begin{aligned} \left[D_x^2 - \lambda\right] Q.R = 0, \\ \left[\kappa D_x^3 + 3\kappa\lambda D_x + \alpha D_x + D_t\right] Q.R = 0, \end{aligned} \tag{10}$$

is also a solution of Eq. (1)*. Consequently, the system of equations* (10) *is referred to as a BBT for Eq.* (1)*.*

Proof: For the get BBT of Eq. (1), we assume that

$$\hat{p} = 2\ln(Q),\ p = 2\ln(R) \tag{11}$$

are two different solutions of (5). Adding two new auxiliary variables

$$\sigma = \frac{\hat{p} + p}{2} = \ln(QR),\ \ \tau = \frac{\hat{p} - p}{2} = \ln\left(\frac{Q}{R}\right), \tag{12}$$

We link the two-field condition by associating it with

$$E(\hat{p}) - E(p) = E(\sigma + \tau) - E(\sigma - \tau) = 12\kappa\sigma_{xx}\tau_{xx} + 2\kappa\tau_{xxxx} + 2\alpha\tau_{xx} + 2\tau_{xt} = 0. \tag{13}$$

The two-field condition (13) must be written into a collection of distinct limitations. In order to do this, an additional restriction needs be added, which would enable Eq. (13) to be reformulated as the $\mathbb{Y}$-derivative of a set of $\mathbb{Y}$-polynomials. Next, we may choose the;

$$\tau_x^2 + \sigma_{xx} = \lambda. \tag{14}$$

which means that $\mathbb{Y}_{0,2x} = \lambda$. Based on the Bell polynomial theory, Eq. (13) can exspressed as

$$E(\hat{p}) - E(p) = 3\kappa\mathbb{Y}_{0,x}(\tau,\sigma)\lambda + \kappa\mathbb{Y}_{0,3x}(\tau,\sigma) + \mathbb{Y}_{0,x}(\tau,\sigma)\alpha + \mathbb{Y}_{t,0}(\tau,\sigma) = 0 \tag{15}$$

Using (14) and (15) we achieve the coupled system of $\mathbb{Y}(\tau,\sigma)$ as follows:

$$\begin{aligned} Y_{0,2x}(v,w) - \lambda &= 0, \\ 3\kappa Y_{0,x}(v,w)\lambda + \kappa Y_{0,3x}(v,w) + Y_{0,x}(v,w)\alpha + Y_{t,0}(v,w) &= 0. \end{aligned} \tag{16}$$

Using the relation between the bilinear operators and $\mathbb{Y}$-polynomials;

$$\mathbb{Y}_{i_1x_1,\ldots,i_mx_m}\Big(\tau = \ln\Big(\frac{Q}{R}\Big), \sigma = \ln(QR) = (QR)^{-1}D_{x_1}^{i_1}\ldots D_{x_m}^{i_m}Q.R\Big), \tag{17}$$

where $i_1 + i_2 + \cdots + i_m \geq 1$. Hence we get the BBT of Eq. (1) as;

$$\begin{aligned} \big[D_x^2 - \lambda\big]Q.R &= 0, \\ \big[\kappa D_x^3 + 3\kappa\lambda D_x + \alpha D_x + D_t\big]Q.R &= 0. \end{aligned}$$

2.3 Lax pair

In this part, we obtain Lax pair of Eq. (1) based on the coupled system of $\mathbb{Y}(\tau,\sigma)$ i.e. Eq. (16) [10], [14].

Theorem 3. *Eq.* (1) *has the following Lax pair:*

$$\begin{aligned} \phi u_x + \phi_{xx} - \lambda\phi &= 0 \\ \kappa(3u_x\phi_x + 3\phi_x\lambda + \phi_{xxx}) + \phi_x\alpha + \phi_t &= 0 \end{aligned} \tag{18}$$

where u is a solution of Eq. (1)*.*

Proof: Using the transformation $\tau = \ln(\phi)$ and Bell polynomial formula

$$\mathbb{Y}_{i_1x_1,\ldots,i_mx_m}(\tau,\sigma)|_{\tau=\ln(Q/R),\sigma=\ln(QR)} = \sum_{q_1=0}^{i_1}\cdots\sum_{q_m=0}^{i_m} C_{i_1}^{q_1}\ldots C_{i_m}^{q_m}P_{q_1x_1,\ldots,q_mx_m}(p)\frac{\phi_{(i_1-q_1)x_1,\ldots(i_m-q_m)x_m}}{\phi}, \tag{19}$$

hence we have

$$\mathbb{Y}_{0,3x}(\tau,\sigma) = \frac{3p_x x\phi_x + \phi_x xx}{\phi}\ ,\ \mathbb{Y}_{0,x}(\tau,\sigma) = \frac{\phi_x}{\phi}\ ,\ \mathbb{Y}_{t,0}(\tau,\sigma) = \frac{\phi_t}{\phi}\ ,\ \mathbb{Y}_{0,2x} = p_{xx} + \frac{\phi_{xx}}{\phi}. \tag{20}$$

Hence the system (16) turns into;

$$\begin{aligned} \phi p_{xx} + \phi_{xx} - \lambda\phi &= 0 \\ \kappa(3p_{xx}\phi_x + 3\phi_x\lambda + \phi_{xxx}) + \phi_x\alpha + \phi_t &= 0 \end{aligned} \tag{21}$$

with the help of (20). Thus we obtain Lax system (18) of Eq. (1) with $u = p_x$ [11].

3 BNNM

In this section introduces the BNNM [18]–[20] and proposes a related tensor formula to provide exact analytical solutions for NLPDEs. To provide exact solutions for Eq. (3), a tensor formula for the nonlinear neural network is developed as [18]:

$$Q = W_{l_n} Q F_{l_n}(\xi_{l_n}), \tag{22}$$

where $W_{a,b}$ denotes the weight coefficient connecting neuron a to neuron b. The function F represents a generalized activation function that must satisfy the condition $F_{l_n}(\xi) \geq 0$ in last layer. Here, the set $l_n = k_{n-1}+1, k_{n-1}+2, \ldots, n$ represents the n-th layer field in the neural network model. The variable ξ_{lj} is defined as follows:

$$\xi_{lj} = W_{lj-1,lj} F_{lj-1}(\xi_{lj-1}) + b_{lj}\,, \; j = 1, 2, \ldots, n. \tag{23}$$

Here $l_0 = x, y, \ldots, t$, $l_1 = 1, 2, \ldots, k_1$, and $l_j = k_{j-1}+1, k_{j-1}+2, \ldots, k_j$ for $j = 2, 3, \ldots, n-1$. The term b represents a threshold value. A diagram is provided to better understand this neural network (Figure 1). Substituting Eq. (22) into Eq. (3) yields an equation. Setting the coefficients of all terms in this equation equal to zero produces a system of algebraic equations. Solving these equations, and substituting these solutions into the transformation given in Eq. (2) along with Eq. (22), allows us to obtain exact analytical solutions for NLPDEs.

3.1 Exact solutions of Eq. (1) via BNNM

To obtain the exact solution of the model, firstly we adopt a "2-2-2-1" neural network architecture. This configuration suggests that there are two neurons in the input layer l_0, in the first hidden layer l_1, in the second hidden layer l_2 and one neuron in the output layer Q. We consider $l_0 = \{x, t\}$, $l_1 = \{1, 2\}$ and $l_2 = \{3, 4\}$, hence we write;

$$\begin{aligned}
Q &= W_{1,Q} F_3(\xi_3) + W_{2,Q} F_4(\xi_4) + b_5,\\
\xi_3 &= W_{1,3} F_1(\xi_1) + W_{2,3} F_2(\xi_2) + b_3,\\
\xi_4 &= W_{1,4} F_1(\xi_1) + W_{2,4} F_2(\xi_2) + b_4,\\
\xi_1 &= W_{x,1} x + W_{t,1} t + b_1,\\
\xi_2 &= W_{x,2} x + W_{t,2} t + b_2.
\end{aligned} \tag{24}$$

In **Case 1** by choosing $F_1 = \cos(\square)$, $F_2 = \exp(\square)$, $F_3 = (\square)^2$, and $F_4 = (\square)$ we reach;

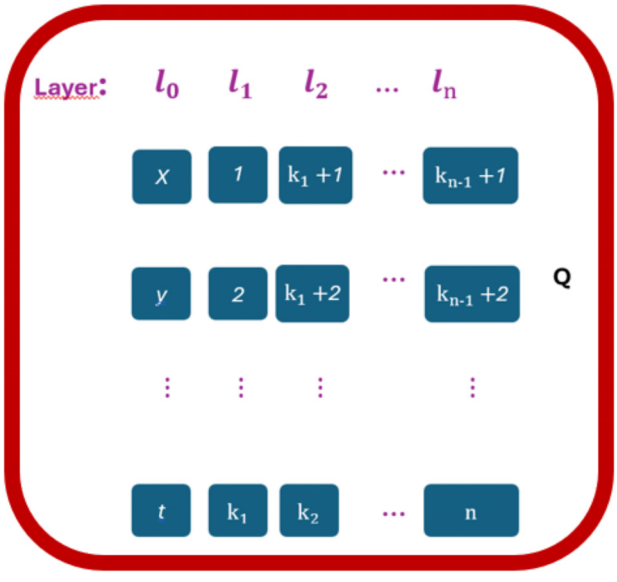

Figure 1: Nonlinear neural network model of Eq. (22): $l_0 = x, y, \ldots, t$, $l_1 = 1, 2, \ldots, k_1$, and $l_j = k_{j-1}+1, k_{j-1}+2, \ldots, k_j$ for $j = 2, 3, \ldots, n-1$.

$$Q = W_{1,Q}\left(W_{1,3}\cos(\xi_1) + W_{2,3}e^{\xi_2} + b_3\right)^2 + W_{2,Q}\left(W_{1,4}\cos(\xi_1) + W_{2,4}e^{\xi_2} + b_4\right) + b_5. \tag{25}$$

Putting (25) into Eq. (3) we obtain two solution sets.

Set 1:

$$\left\{ b_4 = -\frac{b_3^2 W_{1,Q} + b_5}{W_{2,Q}}, W_{1,3} = 0, W_{2,4} = -\frac{2W_{1,Q}W_{2,3}b_3}{W_{2,Q}}, W_{t,1} = 4\kappa W_{x,1}^3 - \alpha W_{x,1}, \right.$$
$$\left. W_{t,2} = -\frac{i}{2}\left(-4\kappa W_{x,1}^2 + \alpha\right)W_{x,1}, W_{x,2} = \frac{i}{2}W_{x,1} \right\} \tag{26}$$

By substituting the values in (26) into Eq. (25) with (2) we yield

$$u_1(x,t) = \frac{6\kappa W_{x,1}\left(ie^{4\,i\kappa t W_{x,1}^3 + i(-\alpha t + x)W_{x,1} + 2b_2}W_{1,Q}W_{2,3}^2 + \sin(S)W_{1,4}W_{2,Q}\right)}{\beta\left(e^{4\,i\kappa t W_{x,1}^3 + i(-\alpha t + x)W_{x,1} + 2b_2}W_{1,Q}W_{2,3}^2 + W_{1,4}\cos(S)W_{2,Q}\right)} \tag{27}$$

where $S = -4\kappa t W_{x,1}^3 + (\alpha t - x)W_{x,1} - b_1$.

Set 2:

$$\left\{ b_5 = -b_3^2 W_{1,Q} - W_{2,Q}b_4, W_{1,3} = 0, W_{2,4} = -\frac{2W_{1,Q}W_{2,3}b_3}{W_{2,Q}}, W_{t,1} = \pm 2\,i\left(16\kappa W_{x,2}^2 + \alpha\right)W_{x,2}, \right.$$
$$\left. W_{t,2} = -16\kappa u_{x,2}^3 - \alpha W_{x,2}, W_{x,1} = \pm 2\,i W_{x,2} \right\} \tag{28}$$

Putting the values in (28) into Eq. (25) with (2) we obtain

$$u_2(x,t) = \frac{12\kappa W_{x,2}\left(e^{-32\kappa t W_{x,2}^3 + (-2\alpha t + 2x)W_{x,2} + 2b_2}W_{1,Q}W_{2,3}^2 + i\,\sin(K)W_{1,4}W_{2,Q}\right)}{\beta\left(e^{-32\kappa t W_{x,2}^3 + (-2\alpha t + 2x)W_{x,2} + 2b_2}W_{1,Q}W_{2,3}^2 + W_{1,4}\cos(K)W_{2,Q}\right)} \tag{29}$$

where $K = 32\,i\kappa t W_{x,2}^3 + 2\,i(\alpha t - x)W_{x,2} - b_1$.

In **Case 2** by choosing $F_1 = \cosh(\square)$, $F_2 = \exp(\square)$, $F_3 = (\square)^3$, and $F_4 = (\square)$ we reach;

$$Q = W_{1,Q}\left(W_{1,3}\cosh(\xi_1) + W_{2,3}e^{-\xi_2} + b_3\right)^3 + W_{2,Q}\left(W_{1,4}\cosh(\xi_1) + W_{2,4}e^{-\xi_2} + b_4\right) + b_5. \tag{30}$$

By following the steps described above, the following set of solutions is obtained.

Set 1:

$$\left\{ b_3 = 0, b_5 = -b_4 W_{2,Q}, W_{1,3} = 0, W_{2,4} = 0, \right.$$
$$\left. W_{t,1} = -108\kappa W_{x,2}^3 - 3\alpha W_{x,2}, W_{t,2} = -36\kappa W_{x,2}^3 - \alpha W_{x,2}, W_{x,1} = 3W_{x,2} \right\} \tag{31}$$

By incorporating the values from (31) into (30) through the use of (2), we arrive;

$$u_3(x,t) = -\frac{18\kappa W_{x,2}\left(W_{1,Q}W_{2,3}^3e^{T-3b_2} + \sinh(T - b_1)W_{1,4}W_{2,Q}\right)}{\beta\left(W_{1,Q}W_{2,3}^3e^{T-3b_2} + \cosh(T - b_1)W_{1,4}W_{2,Q}\right)} \tag{32}$$

where $T = 3W_{x,2}\left(36\kappa W_{x,2}^2 + \alpha\right)t - 3W_{x,2}x$.

4 Conservation laws

Conservation laws are essential for gaining insights into the physical characteristics and interpretations of different systems. In this section, we utilize the multiplier approach, to derive local conservation laws for Eq. (1). Let investigate a m-th order system of partial differential equations with k-independent variables $x = (x^1, x^2, x^3, \ldots, x^k)$ and$n$-dependent variables $\upsilon = (\upsilon^1, \upsilon^2, \ldots, \upsilon^n)$,

$$E^\alpha(x, \upsilon, \upsilon_1, \ldots, \upsilon_m) = 0, \alpha = 1, \ldots, n. \tag{33}$$

Here, $v(i)$ denotes the set of i-th order partial derivatives, where $v_i^\alpha = \mathcal{D}_i(v^\alpha)$, $v_{ij}^\alpha = \mathcal{D}_j\mathcal{D}_i(v^\alpha)$, and so on. The total differentiation operator with respect to x_i is defined as:

$$\mathcal{D}_i = \frac{\partial}{\partial x_i} + v_i^\alpha \frac{\partial}{\partial v^\alpha} + v_{ij}^\alpha \frac{\partial}{\partial v_j^\alpha} + \cdots, \quad i = 1, \ldots, k. \tag{34}$$

The variational derivative operator, denoted as $\frac{\delta}{\delta v}$, is defined by:

$$\frac{\delta}{\delta v^\alpha} = \frac{\partial}{\partial v^\alpha} + \sum_{s\geq 1}(-1)^s \mathcal{D}_{i_1} \ldots \mathcal{D}_{i_s} \frac{\partial}{\partial v_{i_1\ldots i_s}^\alpha}, \quad \alpha = 1, \ldots, n. \tag{35}$$

Definition 4.1. The conserved vector linked to Eq.(33), where each C^i belongs to the collection of all differential functions, satisfies the following relationship:

$$\mathcal{D}_i C^i\Big|_{(33)} = 0 \tag{36}$$

for the solutions of (33).

It is known that the $\Lambda_\alpha(x, v, v_1, \ldots)$ satisfies the following property.

$$\Lambda_\alpha E_\alpha = \mathcal{D}_i C^i \tag{37}$$

In our examination, we will be concentrating on second-order multipliers, specifically represented as $\Lambda_\alpha = \Lambda_\alpha(x, t, v, v_x, v_{xx})$. In Eq.(37) $\mathcal{D}_i C^i$ signifies a divergence condition. To find multipliers, we set up the determining equation for Λ_α:

$$\frac{\delta\left(\Lambda_\alpha E^\alpha\right)}{\delta v^\alpha} = 0. \tag{38}$$

With the help of Eq.(38) we can find all pertinent multipliers, which can subsequently be articulated as local conserved vectors [21]–[25].

Now we obtain the conservation vectors of (1) with the approach we explained in detail above. The identifying equation for the second-order multiplier $\Lambda(x, t, u)$ is

$$\frac{\delta}{\delta u}\left[\Lambda\left(u_t + \beta u_x^2 + \alpha u_x + \kappa u_{xxx}\right)\right] = 0 \tag{39}$$

By expanding the expression (39) and separating it according to different combinations of derivatives of u, an system for the multipliers is obtained. When this system is solved,

$$\Lambda = \left(c_1 t + c_2\right)u_{xx} - \frac{c_1}{2\beta} \tag{40}$$

is obtained. Here c_1, c_2 and c_3 are arbitrary constants. In relation to the aforementioned multipliers, we have the following conserved vectors for Eq. (1).

Case 1:

$$\Lambda_1 = t u_{xx} - \frac{1}{2\beta},$$

$$C_1^t = \frac{\left(-u_x^2\beta - u_x\alpha\right)t - u}{2\beta},$$

$$C_1^x = \frac{2u_x^3 t\beta^2 + 6\left(\frac{1}{2}u_x^2\alpha + \frac{1}{2}u_{xx}^2\kappa + u_t u_x\right)t\beta + 3u_t\alpha t - 3u_{xx}\kappa}{6\beta}.$$

Case 2:

$$\Lambda_2 = u_{xx}$$

$$C_2^t = -\frac{u_x^2}{2},$$

$$C_2^x = \frac{1}{3}u_x^3\beta + \frac{1}{2}u_x^2\alpha + \frac{1}{2}u_{xx}^2\kappa + u_t u_x.$$

5 Discussion and conslusion

The stochastic potential Korteweg–de Vries (spKdV) equation, which appears in the domains of plasma physics and optics, has been examined in this work. The model's bilinear formalism has been established by using the Bell polynomial approach, and the corresponding Bäcklund transformations and Lax pairs have been

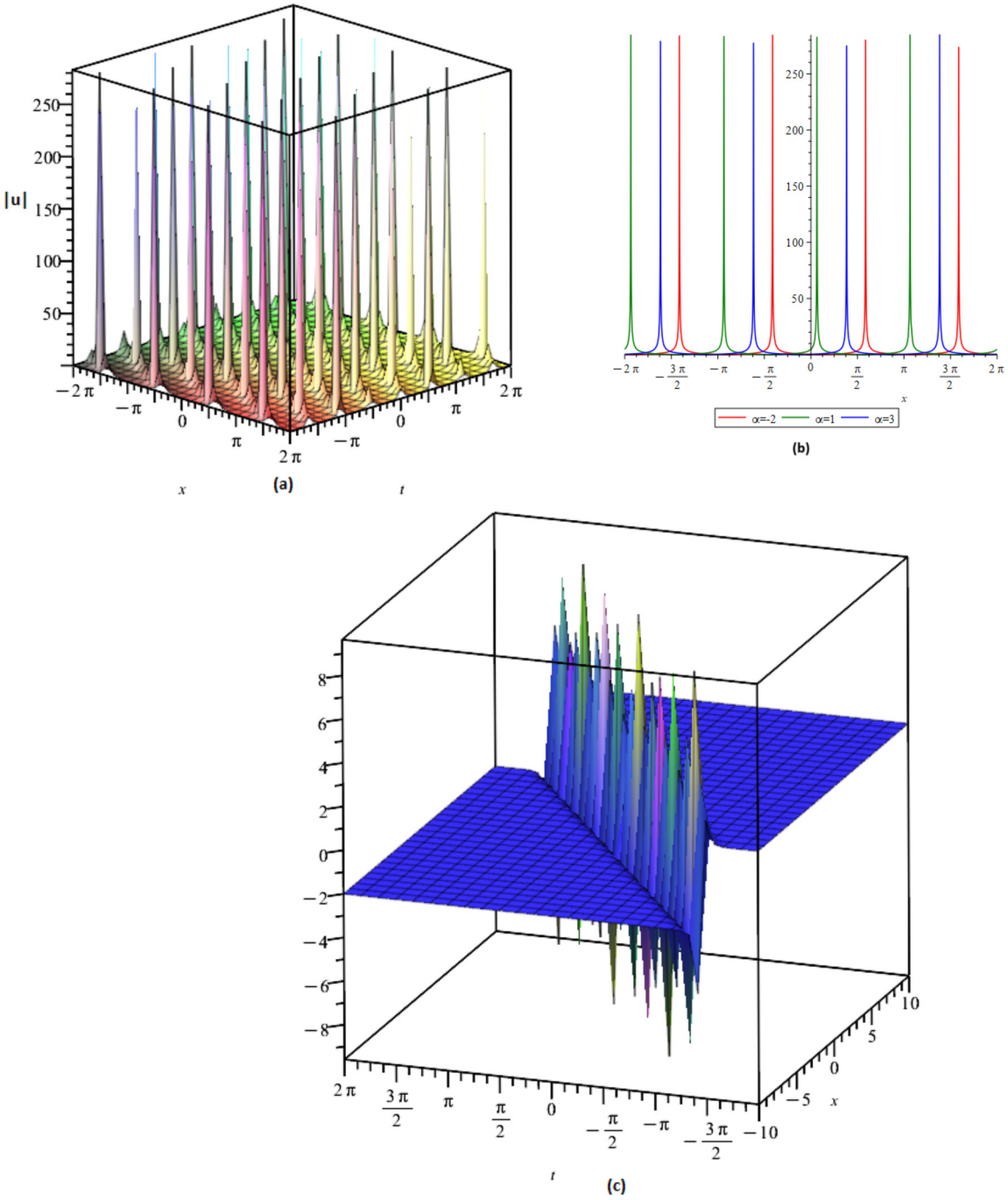

Figure 2: Graphical dynamic analyses of $u_1(x,t)$ and $u_2(x,t)$. (a) 3D-plot of $|u_1(x,t)|$ and (b) 2D-plot of $|u_1(x,t)|$ for changing values of the stochastic parameter α with $\kappa = 0.2$, $b_1 = 1.5$, $W_{x,1} = 1$, $b_2 = -1$, $W_{1,Q} = 1$, $W_{2,3} = 0.5$, $W_{1,4} = 1$, $W_{2,Q} = 1$ and $W_{2,3} = 0.5$ (c) complex plot 3D-plot of $u_2(x,t)$ with $\kappa = 0.2$, $b_2 = -1$, $w_{x,2} = 1$, $b_1 = 1.5$, $W_{1,Q} = 1$, $W_{2,3} = 0.5$, $W_{1,4} = 1$, and $\alpha = -2$.

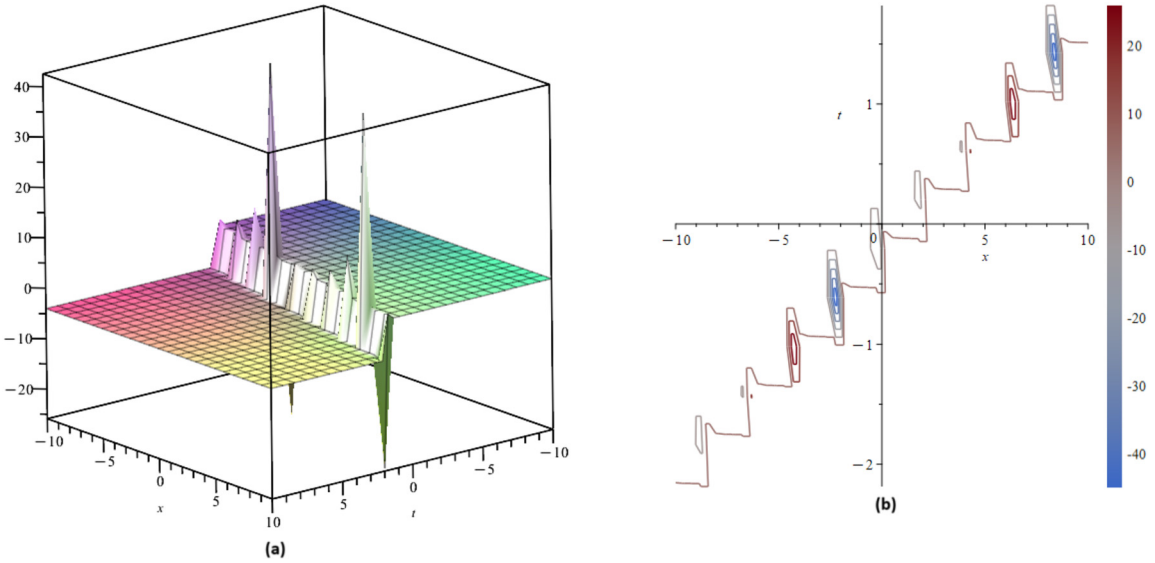

Figure 3: Graphical dynamic analyses of $u_3(x,t)$. (a) 3D-plot of $u_3(x,t)$ and (b) contour-plot of $u_3(x,t)$ with $b_1 = 0.5$, $W_{2,Q} = -0.8$, $b_2 = -2$, $\beta = 0.9$, $W_{1,Q} = 3$, $W_{x,2} = 1$, $\alpha = -2$, $\kappa = 0.2$, $W_{2,3} = 0.9$, and $W_{1,4} = 4$.

methodically derived. This framework offers a straightforward and efficient method for examining the spKdV equation's integrability characteristics. Additionally, the bilinear neural network method (BNNM) has been used as a useful technique to obtain exact analytical solutions of the model inside the bilinear framework. Graphical representations of the obtained solutions have made it possible to clearly see their structural characteristics. In Figure 2(a), the three-dimensional plot of the solution $|u_1(x,t)|$ is presented for the parameter values $\kappa = 0.2$, $b_1 = 1.5$, $w_{x,1} = 1$, $b_2 = -1$, $w_{1,Q} = 1$, $w_{2,3} = 0.5$, $w_{1,4} = 1$, and $w_{2,Q} = 1$. Figure 2(b) illustrates the influence of different values of the stochastic parameter α on the behavior of the solution. Although variations in α do not lead to noticeable changes in the wave type or amplitude, clear shifts in the wave profile are observed, indicating the sensitivity of the solution to stochastic effects. In Figure 2(c), the solution $u_2(x,t)$ corresponding to a dark-bright wave structure is depicted, representing the dynamic interaction between two wave components with distinct energy and intensity characteristics. In Figure 3(a), a three-dimensional plot of the solution $u_3(x,t)$ is shown, while Figure 3(b) presents the corresponding two-dimensional contour plot for the parameter set $b_1 = 0.5$, $w_{2,Q} = -0.8$, $b_2 = -2$, $\beta = 0.9$, $w_{1,Q} = 3$, $w_{x,2} = 1$, $\alpha = -2$, $\kappa = 0.2$, $w_{2,3} = 0.9$, and $w_{1,4} = 4$. These graphical results further demonstrate the effect of the stochastic parameter on the qualitative features of the solutions.

In addition, the conservation vectors of the spKdV equation have been derived using the multiplier method, revealing the inherent conservation structure of the model. The results presented in this work contribute to the theoretical understanding of stochastic nonlinear evolution equations and provide a useful analytical framework that may be extended to other nonlinear models arising in applied mathematics and mathematical physics.

Acknowledgments: This work was supported for 1st author by the Scientific and Technological Research Council of Turkey (TÜBİTAK) under the 2211/A National PhD Scholarship Program.

Research ethics: Not applicable (This study involves mathematical modeling and theoretical analysis; no human or animal subjects were used).

Informed consent: Not applicable (No human participants were involved in this research).

Author contributions: N.G.A conceived the methodology, performed numerical simulations, and manuscript drafting. E.Y. contributed to theoretical framework design and validation. All authors reviewed and approved the final manuscript.

Use of Large Language Models, AI and Machine Learning Tools: None declared (No LLM, AI, or ML tools were used in the design, analysis, or writing of this manuscript).
Conflict of interest: The authors state no conflict of interest.
Research funding: This study was funded by the Scientific and Technological Research Council of Turkey (TÜBİTAK) under grant number 2211/A.
Data availability: Not applicable (This work is theoretical and does not involve raw datasets. All derived results are included in the manuscript).

References

[1] R. K. Gazizov and N. H. Ibragimov, "Lie symmetry analysis of differential equations in finance," *Nonlinear Dyn.*, vol. 17, no. 4, pp. 387—407, 1998.
[2] R. Hirota, *The Direct Method in Soliton Theory (No. 155)*, Cambridge University Press, 2004.
[3] B. Ghanbari and J. F. Gómez-Aguilar, "The generalized exponential rational function method for Radhakrishnan-Kundu-Lakshmanan equation with β-conformable time derivative," *Revista mexicana de física*, vol. 65, no. 5, pp. 503—518, 2019.
[4] J. Zhang, X. Wei, and Y. Lu, "A generalized (G'/G)-expansion method and its applications," *Phys. Lett. A*, vol. 372, no. 20, pp. 3653—3658, 2008.
[5] B. Ghanbari and M. Inc, "A new generalized exponential rational function method to find exact special solutions for the resonance nonlinear Schrödinger equation," *Eur. Phys. J. Plus*, vol. 133, no. 4, p. 142, 2018.
[6] E. Yaşar and E. Yaşar, "Exact wave solutions and local conservation laws of a third-order Boussinesq system," *Discrete Continuous Dyn. Syst.-S*, vol. 18, no. 4, pp. 1075—1092, 2025.
[7] C. Li, L. Chen, and G. Li, "Optical solitons of space-time fractional Sasa-Satsuma equation by F-expansion method," *Optik*, vol. 224, 2020, Art. no. 165527.
[8] N. H. Aljahdaly, S. A. El-Tantawy, A. M. Wazwaz, and H. A. Ashi, "Adomian decomposition method for modelling the dissipative higher-order rogue waves in a superthermal collisional plasma," *J. Taibah Univ. Sci.*, vol. 15, no. 1, pp. 971—983, 2021.
[9] A. J. A. M. Jawad, M. D. Petković, and A. Biswas, "Modified simple equation method for nonlinear evolution equations," *Appl. Math. Comput.*, vol. 217, no. 2, pp. 869—877, 2010.
[10] E. T. Bell, "Exponential polynomials," *Ann. Math.*, vol. 35, no. 2, pp. 258—277, 1934.
[11] J. C. Pu and Y. Chen, "Integrability and exact solutions of the (2+ 1)-dimensional KdV equation with Bell polynomials approach," *Acta Mathematicae Applicatae Sinica, English Series*, vol. 38, no. 4, pp. 861—881, 2022.
[12] F. Lambert, I. Loris, J. Springael, and R. Willer, "On a direct bilinearization method: Kaup's higher-order water wave equation as a modified nonlocal Boussinesq equation," *J. Phys. A: Math. Gen.*, vol. 27, no. 15, p. 5325, 1994.
[13] K. Hosseini, F. Alizadeh, E. Hinçal, M. Ilie, and M. S. Osman, "Bilinear Bäcklund transformation, Lax pair, Painlevé integrability, and different wave structures of a 3D generalized KdV equation," *Nonlinear Dyn.*, vol. 112, no. 20, pp. 18397—18411, 2024.
[14] Q. Miao, Y. Wang, Y. Chen, and Y. Yang, "PDEBellII: A Maple package for finding bilinear forms, bilinear Bäcklund transformations, Lax pairs and conservation laws of the KdV-type equations," *Comput. Phys. Commun.*, vol. 185, no. 1, pp. 357—367, 2014.
[15] R. Alhami and M. Alquran, "Extracted different types of optical lumps and breathers to the new generalized stochastic potential-KdV equation via using the Cole-Hopf transformation and Hirota bilinear method," *Opt. Quant. Electron.*, vol. 54, no. 9, p. 553, 2022.
[16] N. Abbas, A. Hussain, M. B. Riaz, T. F. Ibrahim, F. O. Birkea, and R. A. Tahir, "A discussion on the Lie symmetry analysis, travelling wave solutions and conservation laws of new generalized stochastic potential-KdV equation," *Results Phys.*, vol. 56, 2024, Art. no. 107302.
[17] N. K. Pal, P. Chatterjee, A. M. Wazwaz, and S. Raut, "Lax pairs, infinite conservation laws of stochastic potential Korteweg—de Vries equation and their solutions by using Darboux transformation," *Pramana*, vol. 99, no. 1, p. 5, 2024.
[18] R. F. Zhang and S. Bilige, "Bilinear neural network method to obtain the exact analytical solutions of nonlinear partial differential equations and its application to p-gBKP equation," *Nonlinear Dyn.*, vol. 95, no. 4, pp. 3041—3048, 2019.
[19] R. F. Zhang, M. C. Li, J. Y. Gan, Q. Li, and Z. Z. Lan, "Novel trial functions and rogue waves of generalized breaking soliton equation via bilinear neural network method," *Chaos, Solitons Fractals*, vol. 154, 2022, Art. no. 111692.
[20] R. F. Zhang, S. Bilige, J. G. Liu, and M. Li, "Bright-dark solitons and interaction phenomenon for p-gBKP equation by using bilinear neural network method," *Phys. Scr.*, vol. 96, no. 2, 2020, Art. no. 025224.
[21] M. Kaplan, S. San, and A. Bekir, "On the exact solutions and conservation laws to the Benjamin—Ono equation," *J. Appl. Anal. Comput.*, vol. 8, no. 1, pp. 1—9, 2018.
[22] K. R. Adem and C. M. Khalique, "Exact solutions and conservation laws of a (2+ 1)-Dimensional nonlinear KP-BBM equation," in *Abstract and Applied Analysis*, vol. 2013, Hindawi Publishing Corporation, 2013, p. 791863.

[23] A. F. Cheviakov, "GeM software package for computation of symmetries and conservation laws of differential equations," *Comput. Phys. Commun.*, vol. 176, no. 1, pp. 48—61, 2007.
[24] R. Naz, "Conservation laws for some compacton equations using the multiplier approach," *Appl. Math. Lett.*, vol. 25, no. 3, pp. 257—261, 2012.
[25] R. Naz, "Conservation laws for a complexly coupled KdV system, coupled Burgers' system and Drinfeld—Sokolov—Wilson system via multiplier approach," *Commun. Nonlinear Sci. Numer. Simulat.*, vol. 15, no. 5, pp. 1177—1182, 2010.