Upper Bounds for the Level of Normal Subgroups of Hecke Groups

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Abstract. In [4], Greenberg showed that $n \leq 6t^3$ so that $\mu = nt \leq 6t^4$ for a normal subgroup N of level n and index μ having t parabolic classes in the modular group Γ. Accola, [1], improved these to $n \leq 6t^2$ always and $n \leq t^2$ if Γ/N is not abelian. In this work we generalise these results to Hecke groups. We get results between three parameters of a normal subgroup, i.e. the index μ, the level n and the parabolic class number t. We deal with the case $q = 4$, and then obtain the generalisation to other q. Two main problems here are the calculation of the number of normal subgroups and the determination of the bounds on the level n for a given t.

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INTRODUCTION

Hecke groups $H(\lambda_q)$ are the discrete subgroups of $PSL(2,\mathbf{R})$ generated by two elliptic elements $R(z) = \frac{-1}{z}$ and $S(z) = -\frac{1}{z+\lambda_q}$ where $\lambda_q = 2\cos\frac{\pi}{q}, q \geq 3, q \in \mathbf{N}$, see [2]. As a Fuchsian group, they have the signature $(0;2,q,\infty)$. We let $T = RS$.

For $q = 3$ we obtain the well-known modular group Γ. Therefore the other Hecke groups can be considered as some kind of generalisation of Γ.

We define the level n of a normal subgroup N of $H(\lambda_q)$ as the least natural number so that $T^n = I$ in N. The parabolic class number t of N is defined to be the number of conjugacy classes of ∞. In [4], Greenberg showed that $n \leq 6t^3$ so that $\mu \leq 6t^4$ for a normal subgroup N of level n and index μ having t parabolic classes in Γ. Accola, in [1], improved above inequalities to $n \leq 6t^2$ always, and $n \leq t^2$ if Γ/N is not abelian while studying with finite groups generated by two elements whose orders p,q are coprime, where say, $p < q$, and obtained the following inequalities $n \leq pqt^p,\ \mu \leq pqt^{p+1}$ and $\mu \geq (pq)^{\frac{-1}{p}} n^{1+\frac{1}{p}}$. Using these inequalities, we shall obtain similar results for $H(\lambda_q)$ with q is odd. From now on, by N, we denote a normal subgroup of $H(\lambda_q)$ of level n having t parabolic classes. For a better understanding of notions, see [2].

GENERALISATIONS OF SOME RESULTS CONCERNING μ, n AND t

The numbers μ, n and t are related by the following formula $\mu = n \cdot t$, see [5]. As $H(\lambda_q)$ is generated by 2 elliptic elements of orders 2 and q, above inequalities can be reduced to the following: $n \leq 2qt^2,\ \mu \leq 2qt^3$ and $\mu \geq n\sqrt{\frac{n}{2q}}$. Therefore the index is bounded for a given t. As $H(\lambda_q)$ is finitely generated, we have

Theorem 1 *Let q be odd. Then $H(\lambda_q)$ has only finitely many normal subgroups of finite index having t parabolic classes.*

Proof As $H(\lambda_q)$ is finitely generated, there are only finitely many homomorphisms of it onto finite groups. Each such homomorphism gives a normal subgroup as its kernel. This is true for each t. As t and q are given, $2qt^3$ is finite and the index of a normal subgroup obtained in this way is finite. Therefore the result follows.

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Particularly, if q is an odd prime, it can be shown that $H(\lambda_q)$ has only four normal subgroups of finite index, i.e. $H(\lambda_q)$, $H^2(\lambda_q)$, $H^q(\lambda_q)$ where $H^m(\lambda_q)$ denotes the mth power subgroup generated by the mth powers of the elements of $H(\lambda_q)$, and $H'(\lambda_q)$, denotes the commutator subgroup of $H(\lambda_q)$. Therefore we have

Corollary 2 *If q is an odd prime, then $H(\lambda_q)$ has a normal subgroup of finite index having t parabolic classes if and only if $t = 1$.*

Proof This follows from the signatures of the above four subgroups. Indeed $H(\lambda_q) \cong (0;2,q,\infty)$, $H^2(\lambda_q) \cong (0,q,q,\infty)$, $H^q(\lambda_q) \cong (0,2^{(q)},\infty)$ and also $H(\lambda_q) \cong (\frac{q-1}{2};\infty)$.

Lemma 3 *Let $q \geq 3$ be odd. If N is a normal subgroup of $H(\lambda_q)$ with non-abelian quotient group having t parabolic classes and of level n, then $t > 1$, i.e. there exists no normal subgroup having only one parabolic class with non-abelian quotient, i.e. if $\mu > 2q$, then $t > 1$; i.e. if $t = 1$, then $\mu \leq 2q$ and $H(\lambda_q)/N$ is abelian, i.e. there are only finitely many normal subgroups with $t = 1$.*

Proof We have $\mu = n \cdot t \leq 2qt^3$. But since q is odd and the quotient is non-abelian, $\mu > 2q$. Hence $2q < 2qt^3$ and therefore $t > 1$.

When q is even the situation is different. For example, for $t = 2$, we can always map onto $D_n \cong (2,2,n)$, $n \in \mathbf{N}$, and this gives an infinite class of normal subgroups denoted by $W_n(\lambda_q)$ having 2 parabolic classes: $W_n(\lambda_q) \cong (0;2^{(n)},\infty,\infty)$. This shows that when q is even, $H(\lambda_q)$ does not necessarily have finitely many normal subgroups having t parabolic classes. For some values of t it will be possible to obtain the same result we obtained for odd q, *i.e.* $H(\lambda_q)$ will have finitely many normal subgroups of finite index having t parabolic classes for some values of t.

THE CASE $q = 4$

i. $t = 1$: By the Riemann-Hurwitz formula (RHF), for each $\mu = 4k$, k odd, $H(\sqrt{2})$ has a unique free cycloidal subgroup H of index μ having signature $(\frac{k+1}{2};\infty)$.

Now if $g \neq 1$, then it is shown, in [2], that $H(\sqrt{2})$ has only finitely many normal subgroups of genus g. If $g = 1$, however, $k = 1$ and $\mu = n = 4$. Therefore $H(\sqrt{2})$ has only one normal free cycloidal subgroup of genus 1 with the signature $(1;\infty)$. In [2], it is also shown that $H(\sqrt{2})$ has 4 non-free cycloidal normal subgroups. Therefore we have proved

Theorem 4 *$H(\sqrt{2})$ has only finitely many normal subgroups with only one parabolic class.*

ii. $t = 2$: In this case we have

Theorem 5 *$H(\sqrt{2})$ has infinitely many non-free normal subgroups having 2 parabolic classes.*

Proof We have seen above that by mapping onto the dihedral group $D_n \cong (2,2,n)$ by a homomorphism, we obtain an infinite class of normal subgroups denoted by $W_n(\sqrt{2})$. As this mapping exists for every n, there are infinitely many such subgroups.

Also by mapping onto $(2,4,4k)$ for $k \in \mathbf{N}$, we obtain a free normal subgroup $[4,4k]_{1,1}$ with signature $(k;\infty,\infty)$. Therefore

Theorem 6 *$H(\sqrt{2})$ has infinitely many normal free subgroups having two parabolic classes.*

For a detailed information about the notation, see [2] or [3].

iii. $t = 3$: Let us soppose that there exists a normal subgroup G of $H(\sqrt{2})$ having 3 parabolic classes. Then $H(\sqrt{2})/G$ has the following representation.

$$H(\sqrt{2})/G \cong \langle\, a,b \mid a^2 = b^4 = (ab)^{4m} = I \,\rangle$$

such that the order of G is $12m$ for $m \in \mathbf{N}$. G has a large cyclic subgroup H with index 3 in G, and $H = \langle a,b \rangle$. Let us put $h = ab$. Let $G = H \cup Ha \cup Hb$. Let us now look at the action of b on the cosets. We have either $b \longrightarrow (1)(2)(3)$ which is not possible as $Hb \neq H$, or $b \longrightarrow (1\,2)(3)$. Because of this second possibility b acts as identity, that is $b \in H$. As $b = I$ in G, we have $b^2 = h^{2m}$. Now b commutes with b and h implying b belongs to the center of G. Let $Y = \langle\, b^2 \,\rangle$. As Y is central it is normal in G. Then the quotient group G/Y is generated by Ya and Yb satisfying the relations $(Ya)^2 = (Yb)^2 = (Yab)^{2m} = I$. Therefore G/Y is either dihedral or a quotient of it. Hence $\mid G/Y \mid \leq 4m$, and therefore $\mid G \mid \leq 8m$. But this is a contradiction as we have supposed $\mid G \mid = 12m$. Hence

Theorem 7 *There exists no normal subgroups with* 3 *parabolic classes in* $H(\sqrt{2})$.

iv. $t = 4$: Then $\mu = 4n$. In this case if R goes to identity then we have finitely many cyclic quotients of $H(\sqrt{2})$, (see [2]). Therefore we only consider the case where R is mapped to an element of order two. Also T is mapped to an element of order n. Now as $S^4 = I$, we have three possibilities for S. First, if S goes to identity, we again obtain cyclic quotients. Secondly, if S goes to an element of order 2, then by RHF, it can be seen that we do not get any normal subgroups in that case. Therefore the only case is that S is mapped to an element of order 4. Then by the permutation method (see [7] or [2]), we obtain the signature of N as $(g;\infty^{(4)})$. By RHF, $g = \frac{n}{2} - 1$. Therefore n is even. If $n = 2$, then N is $K(0)$ which is a member of an infinite class of normal subgroups $K(g)$ of genus g of $H(\sqrt{2})$, see [6], [7] and [2]. If $n = 4$, then $N \cong K(1)$. If $n = 6$ or 8 then we obtain $K(2)$ or $K(3)$, respectively.

Theorem 8 $H(\sqrt{2})$ *has infinitely many normal subgroups* $K(g)$ *with signature* $(g;\infty^{(4)})$ *with 4 parabolic classes. Note that these are free.*

Corollary 9 $H(\sqrt{2})$ *has a normal (free) subgroup with index* μ *having 4 parabolic classes for every* $\mu = 4(2g+1)$.

v. $t = 5$: Then $\mu = 5n$. As we can not map $H(\sqrt{2})$ onto a dihedral group such that the obtained subgroup will have $t = 5$, we have $4 \mid \mu$ and $5 \mid \mu$, and therefore $20 \mid \mu$. The Riemann-Hurwitz formula implies that $g = \frac{5n-12}{8}$. Therefore to have a non-negative genus, we must have n to be an odd multiple of 4. Let $n = 4k$, $k \in \mathbf{N}$, k odd. Then the genus is $g = \frac{5k-3}{2}$. Let, first, $n = 4$. Then $g = 1$ and we obtain $[4,4]_{2,1}$ and $[4,4]_{1,2}$ as normal subgroups. Secondly, consider $n = 12$. In this case if there were a normal subgroup of $H(\sqrt{2})$ then it must be of genus 6, and it is easy to check that there is no such normal subgroup.

For the higher values of n, the index gets quite big making the calculations longer. It seems, for some values of n, that $H(\sqrt{2})$ has normal subgroups with 5 parabolic classes, and for some, it does not. However if there is such a subgroup then the following is satisfied:

Theorem 10 *If N is a normal subgroup of* $H(\sqrt{2})$ *having 5 parabolic classes, then the level n of N must be an odd multiple of 4.*

vi. Let finally $t = 6$. Then $\mu = 6n$. Again there are several possibilities. If R or S goes to identity, this gives us a cyclic quotient of $H(\sqrt{2})$ and by mapping onto this quotient we obtain a genus 0 normal subgroup (for details, see [2]). Therefore we omit this case. Therefore R is mapped to an element of order 2.

For T, there is only one possibility; that is, it must be mapped to an element of order n. There are 2 possibilities for S: either it can be mapped to an element of order 2, giving a dihedral quotient of $H(\sqrt{2})$ which is impossible as we can see by RHF, or it can be mapped to an element of order 4. The RHF implies that $g = \frac{3n}{4} - 2$ for a normal subgroup of $H(\sqrt{2})$ having 6 parabolic classes. Then

Theorem 11 *If N is a normal subgroup of* $H(\sqrt{2})$ *having 6 parabolic classes, then the level n must be divisible by 4.*

It is easy to see that there is no such normal subgroup if $n = 4$ or 8.

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