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RESEARCH PAPER

A New Class of Salagean-Type Multivalent Harmonic Functions Defined by Subordination

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Abstract
Utilizing the concepts of subordination we have introduced a generalized class Salagean-Type of complex-valued multivalent harmonic functions. We construct some properties of our class. The results obtained here include a number of known and new results as their special cases.

Keywords Harmonic functions · Multivalent functions · Salagean differential operator · Subordination

Mathematics Subject Classification 30C45 · 30C80

1 Introduction, Main Notations and Definitions

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain $D \subset \mathbb{C}$ is said to be harmonic in D if u and v are both real harmonic in D . Consider the functions U and V analytic in D so that $u = ReU$ and $v = ImV$. Then the harmonic function f can be expressed by

$$f(z) = h(z) + \overline{g(z)} \quad (z \in D),$$

where $h = (U + V)/2$ and $g = (U - V)/2$.

In particular, h is called the analytic part and g is called the co-analytic part of f. It is known (see Clunie and Sheil-Small 1984) that the function $f = h + \overline{g}$ is locally univalent and sense-preserving in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ if and only if $|g'(z)| < |h'(z)|$ $(z \in D)$. The coefficient estimations, distortion theorems, integral expressions, Jacobi estimates and growth condition in geometric properties of covering theorem of the co-analytic part can be obtained by using the analytic part of harmonic functions (see Ahuja 2014; Ahuja et al. 2019; Aouf and Seoudy 2020; Baksa et al. 2020; Dixit et al. 2011 and the bibliography therein).

For a fixed positive integer $p \geq 1$, let $SH(p)$ denote the class of all multivalent harmonic functions $f = h + \overline{g}$ which are sense-preserving in the open unit disk $\mathbb{U}$ and are of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n + \sum_{n=p}^{\infty} \overline{b_n z^n} \quad (|b_p| < 1). \tag{1}$$

Recent interest in the study of multivalent harmonic function prompted the publication of several articles such as Ahuja and Jahangiri (2001, 2002), Jahangiri et al. (2003, 2009), Yaşar and Yalçın (2011, 2013, 2015) and the references cited therein.

We say that a function $f \in SH(p)$ is subordinate to a function $F \in SH(p)$, and write $f(z) \prec F(z)$, if there exists a complex-valued function w which maps $\mathbb{U}$ into oneself with $w(0) = 0$, such that $f(z) = F(w(z)) \quad (z \in \mathbb{U})$.

Further, for functions $f_1, f_2 \in SH(p)$ of the forms:

$$f_t(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n + \sum_{n=p}^{\infty} \overline{b_n z^n} \quad (t = 1, 2),$$

we define convolution or Hadamard product of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{n=p+1}^{\infty} a_{1,n} a_{2,n} z^n + \sum_{n=p}^{\infty} \overline{b_{1,n} b_{2,n} z^n}.$$

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Note that the class $SH(p)$ for $p = 1$ was defined and studied by Clunie and Sheil-Small (1984).

We propose for the beginning a generalized differential operator as follows.

The differential operator D^k $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ was introduced by Salagean (1983). On the other hand, for $f = h + \bar{g}$ given by (1), (Jahangiri et al. 2009) define the modified Salagean operator D_p^k as

$$D_p^k f(z) = D_p^k h(z) + (-1)^k \overline{D_p^k g(z)} \quad (p \geq 1, k \in \mathbb{N}_0), \tag{2}$$

where

$$D_p^k h(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{n}{p}\right)^k a_n z^n \quad \text{and}$$

$$D_p^k g(z) = \sum_{n=p}^{\infty} \left(\frac{n}{p}\right)^k b_n z^n.$$

Associated with the modified Salagean operator D_p^k, we define a subclass $SH_{p,k}(\alpha, A, B)$ of $SH(p)$ consisting of functions f of the form (1) that satisfy the subordination relation

$$\frac{z\left(D_p^k f(z)\right)_z - \bar{z}\left(D_p^k f(z)\right)_{\bar{z}}}{D_p^k f(z)} \prec \frac{p + [pB + (p-\alpha)(A-B)]z}{1 + Bz}, \tag{3}$$

where $-B \leq A < B \leq 1$ and $0 \leq \alpha < 1$.

Remark 1 For suitably specializing the parameters, the classes $SH_{p,k}(\alpha, A, B)$ reduce to the various subclasses of harmonic univalent functions previously mentioned. Thus, we have the following special cases:

(1) $SH_{1,k}(\alpha, A, B) = SH(k, \alpha, A, B)$ (Altınkaya et al. 2018),
(2) $SH_{1,k}(0, A, B) = H_k(A, B)$ (Dziok 2019),
(3) $SH_{1,0}(0, A, B) = S_H^*(A, B)$ (Dziok 2015),
(4) $SH_{p,k}(0, 2\alpha - 1, 1) = H_p(k+1, k, \alpha)$ (Jahangiri et al. 2009).
(5) $SH_{p,1}(0, 2\alpha - 1, 1) = K_H(p, \alpha)$ (Ahuja and Jahangiri 2001),
(6) $SH_{p,0}(0, 2\alpha - 1, 1) = S_H^*(p, \alpha)$ (Ahuja and Jahangiri 2001; Jahangiri 1999),
(7) $SH_{1,k}(0, 2\alpha - 1, 1) = H(k, \alpha)$ (Jahangiri et al. 2002),
(8) $SH_{1,1}(0, 2\alpha - 1, 1) = K_H(\alpha)$ (Jahangiri 1999),
(9) $SH_{1,0}(0, 2\alpha - 1, 1) = S_H^*(\alpha)$ (Jahangiri 1999),
(10) $SH_{1,1}(0, -1, 1) = K_H$ (Silverman and Silvia 1999),
(11) $SH_{1,0}(0, -1, 1) = S_H^*$ (Silverman 1998).

Further, we define the class $\overline{SH}_{p,k}(\alpha, A, B) \equiv SH_{p,k}(\alpha, A, B) \cap SH_k(p)$, where $SH_k(p)(p \geq 1)$ denote the class of functions $f = h + \overline{g}$ in $SH(p)$ so that h and g are of the form

$$h(z) = z^p - \sum_{n=p+1}^{\infty} |a_n| z^n \quad \text{and}$$
$$g(z) = (-1)^k \sum_{n=p}^{\infty} |b_n| z^n, |b_p| < 1. \tag{4}$$

Using the idea of Dziok and Dziok et al. (see Dziok 2015, 2019; Dziok et al. 2016), in this paper, we find necessary and sufficient conditions, distortion bounds, compactness and extreme points for the above defined class $\overline{SH}_{p,k}(\alpha, A, B)$.

2 The Main Results

In this section we formulate some important results beginning with necessary and sufficient conditions for multivalent harmonic functions in the class $SH_{p,k}(\alpha, A, B)$.

Theorem 2 *Let $f \in SH(p)$. Then $f \in SH_{p,k}(\alpha, A, B)$ if and only if*

$$D_p^k f(z) * \varphi(z; \zeta) \neq 0 \qquad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{U}\backslash\{0\}),$$

where

$$\varphi(z;\zeta) = z^p \frac{(p-\alpha)(B-A)\zeta + (1 + [B + (p-\alpha)(A-B)]\zeta)z}{(1-z)^2}$$
$$- \overline{z}^p \frac{2p + [2pB + (p-\alpha)(A-B)]\zeta + [(1-2p)(1+B\zeta) + (p-\alpha)(B-A)\zeta]\overline{z}}{(1-\overline{z})^2}.$$

Proof Let $f \in SH(p)$ be of the form (1). Then $f \in SH_{p,k}(\alpha, A, B)$ if and only if it satisfies (3) or equivalently

$$\frac{z\left(D_p^k f(z)\right)_z - \bar{z}\left(D_p^k f(z)\right)_{\bar{z}}}{D_p^k f(z)} \neq \frac{p + [pB + (p-\alpha)(A-B)]\zeta}{1 + B\zeta}, \tag{5}$$

where $\zeta \in \mathbb{C}$, $|\zeta| = 1$ and $z \in \mathbb{U}\backslash\{0\}$. Now as

$$D_p^k f(z) = D_p^k f(z) * \left(\frac{z^p}{1-z} + \frac{\overline{z^p}}{1-\bar{z}}\right)$$

and

$$z\Big(D_p^k f(z)\Big)_z - \bar{z}\Big(D_p^k f(z)\Big)_{\bar{z}}$$
$$= D_p^k f(z) * \left(\frac{z^p[p+(1-p)z]}{(1-z)^2} - \frac{\overline{z^p}[p+(1-p)\bar{z}]}{(1-\bar{z})^2}\right),$$

the inequality (5) yields

$$\begin{aligned}
&(1+B\zeta)\left\{z\Big(D_p^k f(z)\Big)_z - \bar{z}\Big(D_p^k f(z)\Big)_{\bar{z}}\right\} \\
&\quad - \{p+[pB+(p-\alpha)(A-B)]\zeta\}D_p^k f(z) \\
&= D_p^k h(z) * \left\{(1+B\zeta)\frac{z^p[p+(1-p)z]}{(1-z)^2}\right. \\
&\quad \left. -(p+[pB+(p-\alpha)(A-B)]\zeta)\frac{z^p}{1-z}\right\} \\
&\quad - (-1)^k\overline{D_p^k g(z)} * \left\{(1+B\zeta)\frac{\overline{z^p}[p+(1-p)\bar{z}]}{(1-\bar{z})^2}\right. \\
&\quad \left. +(p+[pB+(p-\alpha)(A-B)]\zeta)\frac{\overline{z^p}}{1-\bar{z}}\right\} \\
&= D_p^k f(z) * \varphi(z;\zeta) \neq 0.
\end{aligned}$$

This completes the proof. □

A sufficient coefficient bound for the class $SH_{p,k}(\alpha, A, B)$ is provided in the following.

Theorem 3 *Let $f = h + \bar{g}$ be so that h and g are given by (1). Then $f \in SH_{p,k}(\alpha, A, B)$, if*

$$\sum_{n=p}^{\infty}\left(\frac{n}{p}\right)^k [\Phi_n|a_n| + \Psi_n|b_n|] \leq 2(p-\alpha)(A-B), \tag{6}$$

with

$$\Phi_n = (1+B)(n-p) + (p-\alpha)(B-A) \tag{7}$$

and

$$\Psi_n = (1+B)(n+p) - (p-\alpha)(B-A). \tag{8}$$

Proof For this purpose, we need to show that if (6) holds, then $f \in SH_{p,k}(\alpha, A, B)$. By definition of subordination, $f \in SH_{p,k}(\alpha, A, B)$ if and only if there exists a complex valued function w; $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$ such that

$$\frac{z\Big(D_p^k f(z)\Big)_z - \bar{z}\Big(D_p^k f(z)\Big)_{\bar{z}}}{D_p^k f(z)} = \frac{p+[pB+(p-\alpha)(A-B)]w(z)}{1+Bw(z)}$$

or, alternatively

$$\left|\frac{z\Big(D_p^k f(z)\Big)_z - \bar{z}\Big(D_p^k f(z)\Big)_{\bar{z}} - pD_p^k f(z)}{B\left[z\Big(D_p^k f(z)\Big)_z - \bar{z}\Big(D_p^k f(z)\Big)_{\bar{z}}\right] - [pB+(p-\alpha)(A-B)]D_p^k f(z)}\right| < 1.$$

Indeed, letting $|z| = r(0<r<1)$ we obtain

$$\begin{aligned}
&\left|z\Big(D_p^k f(z)\Big)_z - \bar{z}\Big(D_p^k f(z)\Big)_{\bar{z}} - pD_p^k f(z)\right| \\
&\quad - \left|B\left[z\Big(D_p^k f(z)\Big)_z - \bar{z}\Big(D_p^k f(z)\Big)_{\bar{z}}\right]\right. \\
&\quad \left. -[pB+(p-\alpha)(A-B)]D_p^k f(z)\right| \\
&= \left|\sum_{n=p+1}^{\infty}\left(\frac{n}{p}\right)^k (n-p)a_n z^n\right. \\
&\quad \left. +(-1)^{k+1}\sum_{n=p}^{\infty}\left(\frac{n}{p}\right)^k (n+p)\overline{b_n z^n}\right| \\
&\quad - \left|(p-\alpha)(B-A)z^p + \sum_{n=p+1}^{\infty}\left(\frac{n}{p}\right)^k\right. \\
&\quad [B(n-p)+(p-\alpha)(B-A)]a_n z^n \\
&\quad + (-1)^{k+1}\sum_{n=p}^{\infty}\left(\frac{n}{p}\right)^k \\
&\quad \left.[B(n+p)+(p-\alpha)(A-B)]\overline{b_n z^n}\right| \\
&\quad \leq \sum_{n=p+1}^{\infty}\left(\frac{n}{p}\right)^k \\
&\quad [(1+B)(n-p)+(p-\alpha)(B-A)]|a_n||z|^n \\
&\quad + \sum_{n=p}^{\infty}\left(\frac{n}{p}\right)^k \\
&\quad [(1+B)(n+p)-(p-\alpha)(B-A)]|b_n||z|^n \\
&\quad - (p-\alpha)(B-A)|z|^p \\
&\quad = |z|^p\left\{\sum_{n=p+1}^{\infty}\left(\frac{n}{p}\right)^k\right. \\
&\quad [(1+B)(n-p)+(p-\alpha)(B-A)]|a_n||z|^{n-p} \\
&\quad + \sum_{n=p}^{\infty}\left(\frac{n}{p}\right)^k \\
&\quad \left.[(1+B)(n+p)-(p-\alpha)(B-A)]|b_n||z|^{n-p}\right. \\
&\quad \left. -(p-\alpha)(B-A)\right\} \\
&\quad < 0,
\end{aligned}$$

by (6).

The harmonic functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{p^k(p-\alpha)(B-A)}{n^k \Phi_n} x_n z^n + \sum_{n=p}^{\infty} \frac{p^k(p-\alpha)(B-A)}{n^k \Psi_n} \overline{y_n z^n}, \tag{9}$$

where

$$\sum_{n=n+p}^{\infty} |x_n| + \sum_{n=p}^{\infty} |y_n| = 1 \text{ and } |x_p| = 1,$$

show that the coefficient bound given by in Theorem 2 is sharp.

Since

$$\sum_{n=p}^{\infty} \left(\frac{n}{p}\right)^k [\Phi_n |a_n| + \Psi_n |b_n|] = (p-\alpha)(B-A) \sum_{n=p}^{\infty} (|x_n| + |y_n|) = 2(p-\alpha)(B-A),$$

the functions of the form (9) are in $SH_{p,k}(\alpha, A, B)$. □

Now, we show that the bound (6) is also necessary for $\overline{SH}_{p,k}(\alpha, A, B)$.

Theorem 4 *Let $f = h + \overline{g}$ with h and g of the form (4). Then $f \in \overline{SH}_{p,k}(\alpha, A, B)$ if and only if the condition (6) holds.*

Proof In view of Theorem 3, it is adequate to show that $f \notin \overline{SH}_{p,k}(\alpha, A, B)$ if condition (6) does not hold. We note that a necessary and sufficient condition for $f = h + \overline{g}$ given by (4) to be in the class $\overline{SH}_{p,k}(\alpha, A, B)$ is that the coefficient condition (6) to be satisfied. Equivalently, we must have

$$\left|\frac{H(z)}{G(z)}\right| < 1,$$

where

$$H(z) = \sum_{n=p+1}^{\infty} \left(\frac{n}{p}\right)^k (n-p)|a_n| z^n + \sum_{n=p}^{\infty} \left(\frac{n}{p}\right)^k (n+p)|b_n| \overline{z^n}$$

and

$$G(z) = (p-\alpha)(B-A)z^p - \sum_{n=p+1}^{\infty} \left(\frac{n}{p}\right)^k [B(n-p) + (p-\alpha)(B-A)]|a_n| z^n - \sum_{n=p}^{\infty} \left(\frac{n}{p}\right)^k [B(n+p) - (p-\alpha)(B-A)]|b_n| \overline{z^n}.$$

Therefore, putting $z = r < 1$ we obtain

$$\left\{ \sum_{n=p+1}^{\infty} \left(\frac{n}{p}\right)^k (n-p)|a_n| r^{n-p} + \sum_{n=p}^{\infty} \left(\frac{n}{p}\right)^k (n+p)|b_n| r^{n-p} \right\} \times \left\{ (p-\alpha)(B-A) - \sum_{n=p+1}^{\infty} \left(\frac{n}{p}\right)^k [B(n-p) + (p-\alpha)(B-A)]|a_n| r^{n-p} - \sum_{n=p}^{\infty} \left(\frac{n}{p}\right)^k [B(n+p) - (p-\alpha)(B-A)]|b_n| r^{n-p} \right\}^{-1} < 1. \tag{10}$$

If condition (6) does not hold, then condition (10) does not hold for r sufficiently close to 1. Hence, there exists $z_0 = r_0$in (0, 1) for which the quotient (10) is greater than 1. This contradicts the required condition for $f \in \overline{SH}_{p,k}(\alpha, A, B)$, and so the proof is complete. □

Theorem 5 *If $f \in \overline{SH}_{p,k}(\alpha, A, B)$, then for $|z| = r < 1$*

$$|f(z)| \le (1 + |b_p|) r^p + \left(\frac{p}{p+1}\right)^k \frac{(p-\alpha)(B-A) - [p(2+A+B) + \alpha(B-A)]|b_p|}{1 + B + (p-\alpha)(B-A)} r^{p+1}$$

and

$$|f(z)| \ge (1 - |b_p|) r^p - \left(\frac{p}{p+1}\right)^k \frac{(p-\alpha)(B-A) - [p(2+A+B) + \alpha(B-A)]|b_p|}{1 + B + (p-\alpha)(B-A)} r^{p+1}.$$

Proof We only prove the left-hand inequality. The proof for the right-hand inequality is similar and will be omitted. Let $f \in \overline{SH}_{p,k}(\alpha, A, B)$. Taking the absolute value of f we have

$$
\begin{aligned}
|f(z)| &\geq (1-|b_p|)r^p - \sum_{n=p+1}^{\infty}(|a_n|+|b_n|)r^n \\
&\geq (1-|b_p|)r^p - \left(\frac{p}{p+1}\right)^k \\
&\sum_{n=p+1}^{\infty}\left(\frac{n}{p}\right)^k \frac{\Phi_n|a_n|+\Psi_n|b_n|}{1+B+(p-\alpha)(B-A)}r^{p+1} \\
&\geq (1-|b_p|)r^p - \left(\frac{p}{p+1}\right)^k \\
&\frac{(p-\alpha)(A-B)-[p(2+A+B)+\alpha(B-A)]|b_p|}{1+B+(p-\alpha)(B-A)}r^{p+1}.
\end{aligned}
$$

The required outcome is obtained. □

The following covering result follows from the left-hand inequality in Theorem 4.

Corollary 6 *Let $f = h + \overline{g}$ with h and g of the form (4). If $f \in \overline{SH}_{p,k}(\alpha, A, B)$, then*

$$
\left\{ w : |w| < \frac{\lambda(p,k,\alpha,A,B) + \mu(p,k,\alpha,A,B)|b_p|}{1+B+(p-\alpha)(B-A)} \right\} \subset f(\mathbb{U}),
$$

where

$$
\begin{aligned}
\lambda(p,k,\alpha,A,B) &= (p+1)^k[1+B+(p-\alpha)(B-A)] \\
&\quad - p^k(p-\alpha)(B-A)
\end{aligned}
$$

and

$$
\begin{aligned}
\mu(p,k,\alpha,A,B) &= p^k[p(2+A+B)+\alpha(B-A)] \\
&\quad - (p+1)^k[1+B+(p-\alpha)(B-A)].
\end{aligned}
$$

Theorem 7 *Set*

$$
\begin{aligned}
h_p(z) &= z^p, h_n(z) = z^p \\
&\quad - \left(\frac{p}{n}\right)^k \frac{(p-\alpha)(B-A)}{\Phi_n} z^n, \\
&(n = p+1, p+2, \ldots),
\end{aligned}
$$

and

$$
\begin{aligned}
g_n(z) &= z^p \\
&\quad + (-1)^k \left(\frac{p}{n}\right)^k \frac{(p-\alpha)(B-A)}{\Psi_n} \overline{z^n}, \\
&(n = p, p+1, \ldots).
\end{aligned}
$$

Then $f \in \overline{SH}_{p,k}(\alpha, A, B)$ if and only if it can be expressed by

$$
f(z) = \sum_{n=p}^{\infty}(x_n h_n(z) + y_n g_n(z)),
$$

where $x_n \geq 0, y_n \geq 0$ and $\sum_{n=p}^{\infty}(x_n + y_n) = 1$. In particular, the extreme points of $\overline{SH}_{p,k}(\alpha, A, B)$ are $\{h_n\}$ and $\{g_n\}$.

Proof Suppose that

$$
\begin{aligned}
f(z) &= \sum_{n=p}^{\infty}(x_n h_n(z) + y_n g_n(z)) \\
&= \sum_{n=p}^{\infty}(x_n + y_n)z^p - \sum_{n=p+1}^{\infty}\left(\frac{p}{n}\right)^k \frac{(p-\alpha)(B-A)}{\Phi_n} x_n z^n \\
&\quad + \sum_{n=p}^{\infty}\left(\frac{p}{n}\right)^k \frac{(p-\alpha)(B-A)}{\Psi_n} y_n \overline{z^n}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{n=p+1}^{\infty}\left(\frac{n}{p}\right)^k \Phi_n|a_n| + \sum_{n=p}^{\infty}\left(\frac{n}{p}\right)^k \Psi_n|b_n| \\
&= (p-\alpha)(B-A)\left(\sum_{n=p+1}^{\infty} x_n + \sum_{n=p}^{\infty} y_n\right) \\
&= (p-\alpha)(B-A)(1-x_p) \\
&\leq (p-\alpha)(B-A)
\end{aligned}
$$

and so $f \in \overline{SH}_{p,k}(\alpha, A, B)$. Conversely, if $f \in \overline{SH}_{p,k}(\alpha, A, B)$, then

$$
|a_n| \leq \left(\frac{p}{n}\right)^k \frac{(p-\alpha)(B-A)}{\Phi_n} \quad \text{and}
$$

$$
|b_n| \leq \left(\frac{p}{n}\right)^k \frac{(p-\alpha)(B-A)}{\Psi_n}.
$$

Set

$$
x_n = \left(\frac{p}{n}\right)^k \frac{\Phi_n}{(p-\alpha)(B-A)}|a_n| \ (n = p+1, p+2, \ldots)
$$

and

$$
y_n = \left(\frac{p}{n}\right)^k \frac{\Psi_n}{(p-\alpha)(B-A)}|b_n| \ (n = p, p+1, \ldots).
$$

Then note by Theorem 3, we have

$$
0 \leq x_n \leq 1 \ (n = p+1, p+2, \ldots) \text{ and } 0 \leq y_n \leq 1 \ (n = p, p+1, \ldots).
$$

We define

$$
x_p = 1 - \sum_{n=p+1}^{\infty} x_n - \sum_{n=p}^{\infty} y_n
$$

and note that by Theorem 3, $x_p \geq 0$. Consequently, we obtain

$$
f(z) = \sum_{n=p}^{\infty}(x_n h_n(z) + y_n g_n(z)).
$$

The required outcome is derived. □

Next, we show that $\overline{SH}_{p,k}(\alpha, A, B)$ is closed under convex combinations of its members.

Theorem 8 *The class* $\overline{SH}_{p,k}(\alpha, A, B)$ *is closed under convex combination.*

Proof For $m \in \mathbb{N}$, let $f_m \in \overline{SH}_{p,k}(\alpha, A, B)$ be given by

$$f_m(z) = z^p - \sum_{n=p+1}^{\infty} |a_{m,n}| z^n + (-1)^k \sum_{n=p}^{\infty} |b_{m,n}| \overline{z^n}.$$

Then by (6), we get

$$\sum_{n=p}^{\infty} \left(\frac{n}{p}\right)^k \left(\Phi_n |a_{m,n}| + \Psi_n |b_{m,n}|\right) \leq 2(p-\alpha)(B-A).$$

For $\sum_{m=1}^{\infty} \delta_m = 1 (0 \leq \delta_m \leq 1)$, the convex combination of f_m is

$$\sum_{m=1}^{\infty} \delta_m f_m(z) = z^p - \sum_{n=p+1}^{\infty} \left(\sum_{m=1}^{\infty} \delta_m |a_{m,n}|\right) z^n + (-1)^k \sum_{n=p}^{\infty} \left(\sum_{m=1}^{\infty} \delta_m |b_{m,n}|\right) \overline{z^n}.$$

Then on using (10), we can write

$$\begin{aligned} &\sum_{n=p}^{\infty} \left(\frac{n}{p}\right)^k \left(\Phi_n \sum_{m=1}^{\infty} \delta_m |a_{m,n}| + \Psi_n \sum_{m=1}^{\infty} \delta_m |b_{m,n}|\right) \\ &= \sum_{m=1}^{\infty} \delta_m \left(\sum_{n=p}^{\infty} \left(\frac{n}{p}\right)^k \left[\Phi_n |a_{m,n}| + \Psi_n |b_{m,n}|\right]\right) \\ &\leq 2(p-\alpha)(B-A) \sum_{m=1}^{\infty} \delta_m = 2(p-\alpha)(A-B) \end{aligned}$$

and so $\sum_{m=1}^{\infty} \delta_m f_m(z) \in \overline{SH}_{p,k}(\alpha, A, B)$. □

3 Conclusions

The principle of subordination is central to the theory of complex-valued functions. Therefore, using the concepts of subordination, we introduced some new subfamilies of harmonic functions. We investigated some useful problems, including necessary and sufficient conditions, distortion limits, problems with partial sums, convolutions and convexity conditions for the newly defined classes of functions. For these classes, problems like topological properties, integral mean inequalities, and their applications are open for the researchers to determine. Further, in terms of the modified Salagean operator, these problems can be studied for classes of meromorphic type harmonic functions, Bazilevič harmonic functions as well.

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