TÜBİTAK

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 2642 – 2648
© TÜBİTAK
doi:10.3906/mat-1905-88

On congruences for q-analogues of ballot numbers

Sibel KOPARAL*
Department of Mathematics, Faculty of Science and Arts, Kocaeli University, İzmit, Kocaeli, Turkey

Received: 23.05.2019 • **Accepted/Published Online:** 09.09.2019 • **Final Version:** 28.09.2019

Abstract: In this paper, we examine some congruences with q-analogues of ballot numbers. For example, for $n>1$ and $d=0,1,...,n-1$,

$$\sum_{k=1}^{n-d} q^k B_{k,d}^q \equiv -2+(-1)^{n-d}\left(\frac{n-d+1}{3}\right)q^{-\frac{1}{3}\binom{n-d}{2}}-\left(\frac{n-d-1}{3}\right)q^{d+1-\frac{1}{3}\binom{n-d-2}{2}} \pmod{\Phi_n(q)},$$

with the Legendre symbol $\left(\frac{\cdot}{3}\right)$, the q-analogue of ballot number $B_{n,d}^q$, and the nth cyclotomic polynomial $\Phi_n(q)$.

Key words: Congruence, q-binomial coefficient, cyclotomic polynomial, ballot number

1. Introduction

Some elementary combinatorial properties of the Catalan and ballot numbers were given in [2, 6, 7]. The Catalan numbers are used in many mathematical problems [2, 7].

The Catalan numbers are defined by

$$C_n=\frac{1}{n+1}\binom{2n}{n}.$$

In [2], the Catalan numbers are special cases of the ballot numbers

$$B_{n,k}=\frac{k}{2n+k}\binom{2n+k}{n}.$$

The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q=\begin{cases} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0\le k\le n, \\ 0 & \text{otherwise,} \end{cases}$$

where the q-Pochhammer symbol is $(x;q)_n=(1-x)(1-xq)...\left(1-xq^{n-1}\right)$.

*Correspondence: sibel.koparal@kocaeli.edu.tr
2010 *AMS Mathematics Subject Classification:* 11B65, 11A07, 05A10

This work is licensed under a Creative Commons Attribution 4.0 International License.

It is clear that

$$\lim_{q\to 1}\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k},$$

where $\binom{n}{k}$ is the usual binomial coefficient. It satisfies the following identities: for $0 \le k \le n$,

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_q &= q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q, \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k\begin{bmatrix} n-1 \\ k \end{bmatrix}_q, \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= q^{k(n-k)}\begin{bmatrix} n \\ k \end{bmatrix}_{1/q}, \end{aligned} \tag{1.1}$$

and for $1 < n,\ 1 \le d \le n-1$, and $0 \le k \le n+d$,

$$\begin{bmatrix} n+d \\ k \end{bmatrix}_q \equiv \begin{cases} \begin{bmatrix} d \\ k \end{bmatrix}_q & \text{if } 1 \le k \le n+d-1 \\ 1 & \text{otherwise} \end{cases} \pmod{\Phi_n(q)}. \tag{1.2}$$

In [1], Carlitz defined the q-Fibonacci polynomials with the recursion relation

$$F_n^q(t) = F_{n-1}^q(t) + q^{n-2}tF_{n-2}^q(t),$$

where the initial values are $F_0^q(t) = 0,\ F_1^q(t) = 1$. An explicit formula for q-Fibonacci polynomials is

$$F_n^q(t) = \sum_{k\ge 0} q^{k^2}\begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q t^k. \tag{1.3}$$

The q-analogues of the Catalan numbers and the ballot numbers are given as follows:

$$C_{n,q} = \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q\begin{bmatrix} 2n \\ n+1 \end{bmatrix}_q,$$

$$B_{n,d}^q = \frac{[d]_q}{[2n+d]_q}\begin{bmatrix} 2n+d \\ n \end{bmatrix}_q = \begin{bmatrix} 2n+d-1 \\ n \end{bmatrix}_q - q^d\begin{bmatrix} 2n+d-1 \\ n+d \end{bmatrix}_q, \tag{1.4}$$

respectively, where $[n+1]_q = \left(1-q^{n+1}\right)/(1-q)$, $[2n+d]_q = \left(1-q^{2n+d}\right)/(1-q)$, and $[d]_q = \left(1-q^d\right)/(1-q)$.

The n-cyclotomic polynomial $\Phi_n(q)$ is

$$\Phi_n(q) = \prod_{\substack{0\le m<n \\ \gcd(m,n)=1}} \left(q - e^{2\pi mi/n}\right).$$

It is clear from [8] that for $n > 1$,

$$\Phi_n(1) = \begin{cases} p & \text{if } n \text{ is a power of a prime } p, \\ 1 & \text{otherwise,} \end{cases}$$

and prime number p,

$$\prod_{j=1}^{a}\Phi_{p^j}(q)=\frac{1-q^{p^a}}{1-q}.$$

In recent years, there are binomial identities or congruences with q-analogues [3–5, 10]. In [3], Greene gave the following equality conjectured by Krammer:

$$1+2\sum_{k=1}^{n-1}(-1)^k q^{-\binom{k}{2}}\begin{bmatrix}2k-1\\ k\end{bmatrix}_q=\begin{cases}\left(\frac{m}{5}\right)\sqrt{5} & \text{if } n\equiv 0 \pmod 5,\\ \left(\frac{n}{5}\right) & \text{otherwise,}\end{cases}$$

where $q=e^{2\pi mi/n}$ with $0\leq m<n$ integers such that $\gcd(m,n)=1$ and $\left(\frac{\div}{p}\right)$ is the Legendre symbol.

In [4], Guo and Zeng proved some alterations of the Green–Krammer identity including central q-binomial coefficients. For example,

$$\sum_{k=0}^{n-1}(-1)^k q^{-\binom{k+1}{2}}\begin{bmatrix}2k\\ k\end{bmatrix}_q\equiv\left(\frac{n}{5}\right)q^{-\lfloor n^4/5\rfloor} \pmod{\Phi_n(q)}.$$

In [10], Tauraso gave some changes to the Greene–Krammer identity involving q-Catalan numbers. The author gave new identities and some congruences. For example, for $n\in\mathbb{Z}^+$,

$$\sum_{k\geq 0}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n-k\\ k\end{bmatrix}_q=(-1)^n\left(\frac{n+1}{3}\right)q^{\frac{1}{3}\binom{n}{2}}. \tag{1.5}$$

In [9], Ömür and Koparal obtained some congruences for odd prime p,

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}B_{k,d} &\equiv \left(\frac{(p+d-1-\varepsilon)/2}{3}\right)-\left(\frac{(p+d+1+\varepsilon)/2}{3}\right)\\ &\quad-\left(\frac{(d-2+\varepsilon)/2}{3}\right)+\left(\frac{(d+2-\varepsilon)/2}{3}\right)-\delta_{0,d} \pmod p,\end{aligned}$$

and for $p\neq 5$,

$$\sum_{k=0}^{(p-1)/2}(-1)^k B_{k,d}\equiv(-1)^{d+1}\left(\frac{5}{p}\right)L_{d-\left(\frac{5}{p}\right)}-\delta_{0,d} \pmod p,$$

where $d\in\{0,1,2,...,(p-1)/2\}$, $\varepsilon=\frac{1-(-1)^d}{2}$, L_n is the nth Lucas number, and $\delta_{i,j}$ is the Kronecker delta.

In this paper, inspired by the work of Tauraso, we consider some q-analogues of congruences as follows: for $n>1$ and $d=0,1,...,n-1$,

$$\sum_{k=1}^{n-d}q^k B_{k,d}^q\equiv -2+(-1)^{n-d}\left(\frac{n-d+1}{3}\right)q^{-\frac{1}{3}\binom{n-d}{2}}-\left(\frac{n-d-1}{3}\right)q^{d+1-\frac{1}{3}\binom{n-d-2}{2}} \pmod{\Phi_n(q)},$$

and for $d,n\in\mathbb{Z}^+$ such that $d<n-3$,

$$\sum_{k=0}^{n-d}(-1)^k q^{-\binom{k}{2}}B_{k,d}^q\equiv F_{n-d+1}^q\left(q^d\right)-(-1)^d q^{-\binom{d}{2}}\left(F_{n+d+1}^q(1)-F_{d+1}^q(1)-1\right) \pmod{\Phi_n(q)},$$

with the Legendre symbol $\left(\frac{\cdot}{3}\right)$, the q-Fibonacci polynomial $F_n^q(t)$, the q-analogue of ballot number $B_{n,d}^q$, and the nth cyclotomic polynomial $\Phi_n(q)$.

2. On congruences for q-analogues of ballot numbers

In this section, we need some properties for further use.

Lemma 2.1 *For $2<n$, and $0\leq d\leq n$, we have*

$$\begin{bmatrix} 2k+d \\ k \end{bmatrix}_q \equiv \begin{cases} (-1)^k q^{\frac{k(3k+2d+1)}{2}} \begin{bmatrix} n-d-k-1 \\ k \end{bmatrix}_q & \text{if } k=0,1,...,n-d-1 \\ 1 & \text{if } k=n-d \end{cases} \pmod{\Phi_n(q)}, \tag{2.1}$$

and

$$\begin{bmatrix} 2k+d-1 \\ k+d \end{bmatrix}_q \equiv \begin{cases} (-1)^{k+d} q^{\frac{(k+d)(3k+d-1)}{2}} \begin{bmatrix} n-k \\ k+d \end{bmatrix}_q & \text{if } k=0,1,...,n-d-1 \\ 1 & \text{if } k=n-d \end{cases} \pmod{\Phi_n(q)}. \tag{2.2}$$

Lemma 2.2 *For $1<n$ and $1\leq d\leq n-1$, we have*

$$\sum_{k=0}^{d-1} q^{k^2} \begin{bmatrix} n-k+d \\ k \end{bmatrix}_q \equiv F_{d+1}^q(1) \pmod{\Phi_n(q)}. \tag{2.3}$$

Now we first establish the following theorem.

Theorem 2.3 *For $1<n$ and $1\leq d\leq n-1$, we have*

$$\sum_{k=1}^{n-d} q^k \begin{bmatrix} 2k+d-1 \\ k \end{bmatrix}_q \equiv -1+(-1)^{n-d}\left(\frac{n-d+1}{3}\right) q^{-\frac{1}{3}\binom{n-d}{2}} \pmod{\Phi_n(q)}, \tag{2.4}$$

and

$$\sum_{k=1}^{n-d} q^{k+d} \begin{bmatrix} 2k+d-1 \\ k+d \end{bmatrix}_q \equiv 1+(-1)^{n-d}\left(\frac{n-d-1}{3}\right) q^{d+1-\frac{1}{3}\binom{n-d-2}{2}} \pmod{\Phi_n(q)}. \tag{2.5}$$

Proof First, we give the proof of (2.4). By (1.1), we write

$$\begin{aligned} q^k \begin{bmatrix} 2k+d-1 \\ k \end{bmatrix}_q &= q^{k(k+d)} \begin{bmatrix} 2k+d-1 \\ k \end{bmatrix}_{1/q} \\ &= conj\left(q^{-k(k+d)} \begin{bmatrix} 2k+d-1 \\ k \end{bmatrix}_q\right), \end{aligned}$$

where $conj(z)$ is the complex conjugate of $z \in \mathbb{C}$. From (1.5) and (2.1), we rewrite

$$\begin{aligned}\sum_{k=1}^{n-d} q^k \begin{bmatrix} 2k+d-1 \\ k \end{bmatrix}_q &= conj\left(\sum_{k=1}^{n-d} q^{-k(k+d)} \begin{bmatrix} 2k+d-1 \\ k \end{bmatrix}_q\right) \\ &\equiv conj\left(-1+\sum_{k=0}^{\lfloor \frac{n-d}{2} \rfloor} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n-d-k \\ k \end{bmatrix}_q\right) \\ &= -1+(-1)^{n-d}\left(\frac{n-d+1}{3}\right) q^{-\frac{1}{3}\binom{n-d}{2}} \pmod{\Phi_n(q)},\end{aligned}$$

which gives the desired conclusion.

Secondly, the proof of the other congruence is given. From (2.4), we get

$$\sum_{k=2}^{n-d+1} q^{k-1} \begin{bmatrix} 2k+d-3 \\ k-1 \end{bmatrix}_q \equiv -1+(-1)^{n-d}\left(\frac{n-d+1}{3}\right) q^{-\frac{1}{3}\binom{n-d}{2}} \pmod{\Phi_n(q)}. \tag{2.6}$$

Replacing $d+2$ by d in (2.6), we have

$$\sum_{k=2}^{n-d-1} q^{k-1} \begin{bmatrix} 2k+d-1 \\ k+d \end{bmatrix}_q \equiv -1+(-1)^{n-d}\left(\frac{n-d-1}{3}\right) q^{-\frac{1}{3}\binom{n-d-2}{2}} \pmod{\Phi_n(q)}.$$

Thus,

$$\begin{aligned}&\sum_{k=1}^{n-d} q^{k+d} \begin{bmatrix} 2k+d-1 \\ k+d \end{bmatrix}_q \\ \equiv\ & q^{d+1}+q^n \begin{bmatrix} 2n-d-1 \\ n \end{bmatrix}_q - q^{d+1}+(-1)^{n-d}\left(\frac{n-d-1}{3}\right) q^{d+1-\frac{1}{3}\binom{n-d-2}{2}} \\ =\ & q^n \begin{bmatrix} 2n-d-1 \\ n \end{bmatrix}_q +(-1)^{n-d}\left(\frac{n-d-1}{3}\right) q^{d+1-\frac{1}{3}\binom{n-d-2}{2}} \pmod{\Phi_n(q)}.\end{aligned}$$

Since $q^n \equiv 1 \pmod{\Phi_n(q)}$ and (2.2), the proof is completed. □

For example, when $d=3$ in Theorem 2.3, we have

$$\sum_{k=1}^{n-3} q^k \begin{bmatrix} 2k+2 \\ k \end{bmatrix}_q \equiv -1-(-1)^n\left(\frac{n+1}{3}\right) q^{-\frac{1}{3}\binom{n-3}{2}} \pmod{\Phi_n(q)},$$

and

$$\sum_{k=1}^{n-3} q^{k+3} \begin{bmatrix} 2k+2 \\ k-1 \end{bmatrix}_q \equiv 1-(-1)^n\left(\frac{n-1}{3}\right) q^{4-\frac{1}{3}\binom{n-5}{2}} \pmod{\Phi_n(q)}.$$

Theorem 2.4 *For $1<n$ and $0 \le d \le n-1$, we have*

$$\sum_{k=1}^{n-d} q^k B_{k,d}^q \equiv -2+(-1)^{n-d}\left(\frac{n-d+1}{3}\right) q^{-\frac{1}{3}\binom{n-d}{2}} - (-1)^{n-d}\left(\frac{n-d-1}{3}\right) q^{d+1-\frac{1}{3}\binom{n-d-2}{2}} \pmod{\Phi_n(q)}.$$

Proof Using the equality

$$B_{k,d}^{q}=\begin{bmatrix}2k+d-1\\ k\end{bmatrix}_{q}-q^{d}\begin{bmatrix}2k+d-1\\ k+d\end{bmatrix}_{q},$$

(2.4), and (2.5), the desired result is obtained. □

For example, for $d=1$ in Theorem 2.4, we have

$$\sum_{k=0}^{n-1}q^{k}C_{k,q}\equiv\begin{cases}q^{\lfloor n/3\rfloor} & \text{if } n\equiv 0,1 \pmod 3\\ -1-q^{(2n-1)/3} & \text{if } n\equiv 2 \pmod 3\end{cases}\pmod{\Phi_{n}(q)}[10].$$

We are ready to give a special case of Theorem 2.4 by the following corollary.

Corollary 2.5 *Let d,n be integers such that $3\,|n-d-2$. Then*

$$\sum_{k=1}^{n-d}q^{k}B_{k,d}^{q}\equiv -2-(-1)^{n-d}q^{-\frac{1}{3}\binom{d}{2}}\pmod{\Phi_{n}(q)}.$$

Theorem 2.6 *Let d,n be integers such that $d<n-3$. Then*

$$\sum_{k=0}^{n-d}(-1)^{k}q^{-\binom{k}{2}}B_{k,d}^{q}\equiv\ F_{n-d+1}^{q}\left(q^{d}\right)-(-1)^{d}q^{-\binom{d}{2}}\left(F_{n+d+1}^{q}(1)-F_{d+1}^{q}(1)-1\right)\pmod{\Phi_{n}(q)}.$$

Proof With the help of equalities (1.4), (2.1), and (2.2), for $k=0,1,...,n-d-1$, we get

$$\begin{aligned}&(-1)^{k}q^{-\binom{k}{2}}B_{k,d}^{q}\\ =\ &(-1)^{k}q^{-\binom{k}{2}}\begin{bmatrix}2k+d-1\\ k\end{bmatrix}_{q}-(-1)^{k}q^{-\binom{k}{2}+d}\begin{bmatrix}2k+d-1\\ k+d\end{bmatrix}_{q}\\ \equiv\ &q^{k(k+d)}\begin{bmatrix}n-k-d\\ k\end{bmatrix}_{q}-(-1)^{d}q^{-\binom{d}{2}}q^{(k+d)^{2}}\begin{bmatrix}n-k\\ k+d\end{bmatrix}_{q}\pmod{\Phi_{n}(q)},\end{aligned}$$

and then

$$\begin{aligned}&\sum_{k=0}^{n-d}(-1)^{k}q^{-\binom{k}{2}}B_{k,d}^{q}\\ =\ &(-1)^{n-d}q^{-\binom{n-d}{2}}B_{n-d,d}^{q}+\sum_{k=0}^{n-d-1}(-1)^{k}q^{-\binom{k}{2}}B_{k,d}^{q}\\ \equiv\ &\sum_{k\geq 0}q^{k(k+d)}\begin{bmatrix}n-k-d\\ k\end{bmatrix}_{q}-(-1)^{d}q^{-\binom{d}{2}}\sum_{k\geq 0}q^{(k+d)^{2}}\begin{bmatrix}n-k\\ k+d\end{bmatrix}_{q}+(-1)^{d}q^{-\binom{d}{2}}\\ =\ &\sum_{k\geq 0}q^{k(k+d)}\begin{bmatrix}n-k-d\\ k\end{bmatrix}_{q}-(-1)^{d}q^{-\binom{d}{2}}\sum_{k\geq 0}q^{k^{2}}\begin{bmatrix}n-k+d\\ k\end{bmatrix}_{q}\\ &+(-1)^{d}q^{-\binom{d}{2}}\sum_{k=1}^{d-1}q^{k^{2}}\begin{bmatrix}n-k+d\\ k\end{bmatrix}_{q}+2(-1)^{d}q^{-\binom{d}{2}}\pmod{\Phi_{n}(q)}.\end{aligned}$$

By definition (1.3), we have $F_{n-d+1}^{q}\left(q^{d}\right)=\sum_{k\geq 0}q^{k(k+d)}\begin{bmatrix}n-k-d\\ k\end{bmatrix}_{q}$ and $F_{n+d+1}^{q}\left(1\right)=\sum_{k\geq 0}q^{k^{2}}\begin{bmatrix}n-k+d\\ k\end{bmatrix}_{q}$. Hence,

$$\begin{aligned}&\sum_{k=0}^{n-d}\left(-1\right)^{k}q^{-\binom{k}{2}}B_{k,d}^{q}\\ \equiv\;& F_{n-d+1}^{q}\left(q^{d}\right)-\left(-1\right)^{d}q^{-\binom{d}{2}}\left(F_{n+d+1}^{q}\left(1\right)-\sum_{k=1}^{d-1}q^{k^{2}}\begin{bmatrix}n-k+d\\ k\end{bmatrix}_{q}-2\right)\pmod{\Phi_{n}\left(q\right)}.\end{aligned}$$

From (2.3), the desired result is obtained. □

For example, when $d=2$ in Theorem 2.6, we write

$$\sum_{k=1}^{n-2}\left(-1\right)^{k}\frac{q^{-\binom{k}{2}}}{\left[2k+2\right]_{q}}\begin{bmatrix}2k+2\\ k\end{bmatrix}_{q}\equiv\;\frac{1}{\left[2\right]_{q}}F_{n-1}^{q}\left(q^{2}\right)-\frac{1}{q\left[2\right]_{q}}\left(F_{n+3}^{q}\left(1\right)-q-2\right)\pmod{\Phi_{n}\left(q\right)}.$$

Acknowledgement

The author would like to thank the referee for carefully reading the paper and for valuable comments to improve its presentation.

References

[1] Carlitz L. Fibonacci notes 4: q-Fibonacci polynomials. Fibonacci Quarterly 1975; 13: 97-102.

[2] Gessel IM. Super ballot numbers. Journal of Symbolic Computation 1992; 14: 179-194.

[3] Greene J. On a conjecture of Krammer. Journal of Combinatorial Theory Series A 1991; 56: 309-311.

[4] Guo VJW, Zeng J. Some congruences involving central q-binomial coefficients. Advances in Applied Mathematics 2010; 45: 303-316.

[5] Guo VJW, Zudilin W. A q-microscope for supercongruences. Advances in Mathematics 2019; 346: 329-358.

[6] Hilton P, Pedersen J. Catalan numbers, their generalization, and their uses. Mathematical Intelligencer 1991; 13: 64-75.

[7] Hilton P, Pedersen J. The ballot problem and Catalan numbers. Nieuw Archief voor Wiskunde 1990; 8: 209-216.

[8] Nagell T. Introduction to Number Theory. New York, NY, USA: Wiley, 1951.

[9] Ömür N, Koparal S. Some interesting congruences for ballot numbers. Miskolc Mathematical Notes 2018; 19 (2): 1079-1094.

[10] Tauraso R. q-Analogs of some congruences involving Catalan numbers. Advances in Applied Mathematics 2012; 48: 603-614.