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Almost neo balancing numbers

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ABSTRACT

In this work, we defined almost neo balancing numbers and determined the general terms of them in terms of balancing and Lucas-balancing numbers. We also deduced some results on relationship with triangular, square triangular, Pell, Pell-Lucas numbers and these numbers. Further we formulate the sum of first n-terms of these numbers.

Keywords: balancing numbers, cobalancing numbers, Pell equations, triangular and square triangular numbers, set of representations, Pell numbers

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1. Introduction

A positive integer n is called a balancing number [2] if the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r), \tag{1}$$

holds for some positive integer r which is called balancer. If n is a balancing number with balancer r, then from (1)

$$r=\frac{-2n-1+\sqrt{8n^2+1}}{2}. \tag{2}$$

Though the definition of balancing numbers suggests that no balancing number should be less than 2. But from (2), they noted that $8(0)^2+1=1$ and $8(1)^2+1=3^2$ are perfect squares. So they accepted 0 and 1 to be balancing numbers.

Panda and Ray [18] defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$1+2+\cdots+n=(n+1)+(n+2)+\cdots+(n+r), \tag{3}$$

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holds for some positive integer r which is called cobalancer. If n is a cobalancing number with cobalancer r, then from (3)

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 8n + 1}}{2}. \tag{4}$$

From (4), they noted that $8(0)^2 + 8(0) + 1 = 1$ is a perfect square. So they accepted 0 to be a cobalancing number, just like Behera and Panda accepted 0 and 1 to be balancing numbers.

Let B_n denote the balancing number and let b_n denote the cobalancing number. Then from (2), B_n is a balancing number if and only if $8B_n^2 + 1$ is a perfect square and from (4), b_n is a cobalancing number if and only if $8b_n^2 + 8b_n + 1$ is a perfect square. Thus

$$C_n = \sqrt{8B_n^2 + 1} \text{ and } c_n = \sqrt{8b_n^2 + 8b_n + 1},$$

are integers which are called Lucas-balancing number and Lucas-cobalancing number, respectively.

Let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ be the roots of the characteristic equation for Pell and Pell-Lucas numbers which are the numbers defined by $P_0 = 0, P_1 = 1$, $P_n = 2P_{n-1} + P_{n-2}$ and $Q_0 = Q_1 = 2, Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$, respectively. Ray proved [20] that the Binet formulas for all balancing numbers are

$$B_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}, b_n = \frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, C_n = \frac{\alpha^{2n} + \beta^{2n}}{2} \text{ and } c_n = \frac{\alpha^{2n-1} + \beta^{2n-1}}{2},$$

for $n \geq 1$ (see also [7, 13, 19]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [10], Liptai proved that there is no Fibonacci balancing number except 1 and in [11] he proved that there is no Lucas-balancing number. In [23], Szalay considered the same problem and obtained some nice results by a different method. In [9], Kovács, Liptai, Olajos extended the concept of balancing numbers to the (a, b)-balancing numbers defined as follows: Let $a > 0$ and $b \geq 0$ be coprime integers. If

$$(a + b) + \cdots + (a(n - 1) + b) = (a(n + 1) + b) + \cdots + (a(n + r) + b),$$

for some positive integers n and r, then $an+b$ is an (a, b)-balancing number. The sequence of (a, b)-balancing numbers is denoted by $B_m^{(a,b)}$ for $m \geq 1$. In [12], Liptai, Luca, Pintér and Szalay generalized the notion of balancing numbers to numbers defined as follows: Let $y, k, l \in \mathbb{Z}^+$ with $y \geq 4$. A positive integer x with $x \leq y - 2$ is called a (k, l)-power numerical center for y if

$$1^k + \cdots + (x - 1)^k = (x + 1)^l + \cdots + (y - 1)^l.$$

They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for (k, l)-power numerical centers. For positive integers k, x, let

$$\Pi_k(x) = x(x + 1) \ldots (x + k - 1).$$

Then it was proved in [9] that the equation $B_m = \Pi_k(x)$ for fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2, 3, 4\}$ all solutions were determined. In [34], Tengely, considered the case $k = 5$ and proved that this Diophantine equation has no solution for $m \geq 0$ and $x \in \mathbb{Z}$. In [16], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas-balancing numbers. In [21], Ray considered the sums of balancing and Lucas-balancing numbers by matrix methods. In [14], Özdemir introduced a new non-commutative number system called hybrid numbers. In [3], Bród, Szynal-Liana and Włoch defined balancing and Lucas-balancing hybrid numbers and in [22], Rubajczyk and Szynal-Liana defined cobalancing and Lucas-cobalancing hybrid numbers. In [5], Dash, Ota, Dash defined t-balancing numbers. In [27], Tekcan and Aydın determined the general terms of t-balancing and Lucas t-balancing numbers. In [28], Tekcan and Erdem determined the general terms of t-cobalancing and Lucas t -cobalancing numbers.

In [17], Panda and Panda defined almost balancing numbers. They said that a positive integer n is called an almost balancing number if the Diophantine equation

$$|[(n+1) + (n+2) + \cdots + (n+r)] - [1 + 2 + \cdots + (n-1)]| = 1, \tag{5}$$

holds for some positive integer r which is called the almost balancer. In [15], Panda defined almost cobalancing numbers. He said that a positive integer n is called an almost cobalancing number if the Diophantine equation

$$|[(n+1) + (n+2) + \cdots + (n+r)] - (1 + 2 + \cdots + n)| = 1,$$

holds for some positive integer r which is called the almost cobalancer. In [29], Tekcan and Erdem determined the general terms of all almost balancing numbers and almost cobalancing numbers. In [25], Tekcan considered the sums and spectral norms of all almost balancing numbers and in [24], Tekcan derived some results on almost balancing numbers, triangular numbers and square triangular numbers. In [30, 32], Tekcan and Yıldız defined balcobalancing numbers and in [31, 33], they defined almost balcobalancing numbers. In [26], Tekcan and Akg üç defined almost neo cobalancing numbers.

2. Almost neo balancing numbers

In [4], Chailangka and Pakapongpun defined neo balancing numbers. They said that a positive integer n is called a neo balancing number if the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n-1) + (n-0) + (n+1) + (n+2) + \cdots + (n+r), \tag{6}$$

holds for some integer r which is called neo balancer corresponding to n. For example $2, 7, 36, 205$ are neo balancing numbers with neo balancers $-1, 1, 13, 83$, respectively.

By considering (5) and (6), we say that a positive integer n is called an almost neo balancing number if the Diophantine equation

$$\left| \begin{array}{c} (n-1) + (n-0) + (n+1) + (n+2) + \cdots + (n+r) \\ -[1 + 2 + \cdots + (n-1)] \end{array} \right| = 1, \tag{7}$$

holds for some positive integer r which is called almost neo balancer. From (7), we have two cases:

(1) If $(n-1)+(n-0)+(n+1)+(n+2)+\cdots+(n+r)-[1+2+\cdots+(n-1)]=1$, then n is called an almost neo balancing number of first type, r is called an almost neo balancer of first type and in this case

$$r=\frac{-2n-1+\sqrt{8n^2-16n+17}}{2}. \tag{8}$$

For example $19, 106, 613$ are almost neo balancing numbers of first type with almost neo balancers of first type are $6, 42, 252$, respectively.

From (8), we note that for $n=1$ and $n=4$, $8(1)^2-16(1)+17=3^2$ and $8(4)^2-16(4)+17=9^2$ are perfect squares, but in this case $r=0$ and $r=0$ again. Nevertheless we accept 1 and 4 to be almost neo balancing numbers of first type, just like Behera and Panda accepted 0 and 1 to be balancing numbers.

Let B_n^{neo*} denote the almost neo balancing number of first type. Then from (8), B_n^{neo*} is an almost neo balancing number of first type if and only if $8(B_n^{neo*})^2-16B_n^{neo*}+17$ is a perfect square. Thus

$$C_n^{neo*}=\sqrt{8(B_n^{neo*})^2-16B_n^{neo*}+17}, \tag{9}$$

is an integer which is called an almost Lucas-neo balancing number of first type (We denote the almost neo balancer of first type by R_n^{neo*}).

(2) If $(n-1)+(n-0)+(n+1)+(n+2)+\cdots+(n+r)-[1+2+\cdots+(n-1)]=-1$, then n is called an almost neo balancing number of second type, r is called an almost neo balancer of second type and in this case

$$r=\frac{-2n-1+\sqrt{8n^2-16n+1}}{2}. \tag{10}$$

For example $12, 24, 65$ are almost neo balancing numbers of second type with almost neo balancers of second type are $3, 8, 25$, respectively.

From (10), we note that for $n=2,3$ and $n=5$, $8(2)^2-16(2)+1=1^2$, $8(3)^2-16(3)+1=5^2$ and $8(5)^2-16(5)+1=11^2$ are perfect squares, but in this case $r=-2,-1$ and $r=0$, respectively. Nevertheless we accept $2, 3$ and 5 to be almost neo balancing numbers of second type.

Let B_n^{neo**} denote the almost neo balancing number of second type. Then from (10), B_n^{neo**} is an almost neo balancing number of second type if and only if $8(B_n^{neo**})^2-16B_n^{neo**}+1$ is a perfect square. Thus

$$C_n^{neo**}=\sqrt{8(B_n^{neo**})^2-16B_n^{neo**}+1}, \tag{11}$$

is an integer which is called an almost Lucas-neo balancing number of second type. (We denote the almost neo balancer of second type by R_n^{neo**}).

In this paper, we try to determine the general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first and second type in terms of balancing and Lucas-balancing numbers.

2.1. *Almost neo balancing numbers of first type*

We see that B_n^{neo*} is an almost neo balancing number of first type if and only if $8(B_n^{neo*})^2 - 16B_n^{neo*} + 17$ is a perfect square. So we set

$$8(B_n^{neo*})^2 - 16B_n^{neo*} + 17 = y^2,$$

for some positive integer y. Then

$$2[4(B_n^{neo*})^2 - 8B_n^{neo*}] + 17 = y^2,$$

and hence

$$2(2B_n^{neo*} - 2)^2 + 9 = y^2.$$

Taking $x = 2B_n^{neo*} - 2$, we get the Pell equation [1, 8]

$$2x^2 - y^2 = -9. \tag{12}$$

Let Ω^* denotes the set of all integer solutions of (12), that is,

$$\Omega^* = \{(x, y) : 2x^2 - y^2 = -9\}.$$

Then we can give the following theorem.

Theorem 2.1. *The set of all integer solutions of* (12) *is* $\Omega^* = \{(6B_n, 3C_n) : n \geq 1\}$.

Proof. For the Pell equation in (12), the indefinite form is $F = (2, 0, -1)$ of discriminant $\Delta = 8$. So $\tau_8 = 3+2\sqrt{2}$. Thus the set of representatives (see [6, p. 121]) is Rep $= \{[0 \quad 3]\}$ and $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Here we notice that $[0 \quad 3]M^n$ generates all integer solutions (x_n, y_n) for $n \geq 1$. It can be easily seen that the nth power of M is

$$M^n = \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix},$$

for $n \geq 1$. So

$$[x_n \quad y_n] = [0 \quad 3]M^n = [6B_n \quad 3C_n].$$

Thus the set of all integer solutions is $\Omega^* = \{(6B_n, 3C_n) : n \geq 1\}$. □

From Theorem 2.1, we can give the following result.

Theorem 2.2. *The general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first type are*

$$B_n^{neo*} = 3B_{n-1} + 1,\ C_n^{neo*} = 3C_{n-1} \quad \textit{and} \quad R_n^{neo*} = \frac{-6B_{n-1} + 3C_{n-1} - 3}{2},$$

for $n \geq 1$.

Proof. We proved in Theorem 2.1 that the set of all integer solutions of (12) is $\Omega^* = \{(6B_n, 3C_n) : n \geq 1\}$. Since $x = 2B_n^{neo*} - 2$, we get

$$B_n^{neo*} = \frac{6B_{n-1} + 2}{2} = 3B_{n-1} + 1,$$

for $n \geq 1$. Thus from (9), we obtain

$$\begin{aligned} C_n^{neo*} &= \sqrt{8(B_n^{neo*})^2 - 16B_n^{neo*} + 17} \\ &= \sqrt{8(3B_{n-1} + 1)^2 - 16(3B_{n-1} + 1) + 17} \\ &= \sqrt{9(8B_{n-1}^2 + 1)} \\ &= 3C_{n-1}, \end{aligned}$$

for $n \geq 1$. From (8), we deduce that

$$R_n^{neo*} = \frac{-2(3B_{n-1} + 1) - 1 + 3C_{n-1}}{2} = \frac{-6B_{n-1} + 3C_{n-1} - 3}{2},$$

for $n \geq 1$. □

2.2. Almost neo balancing numbers of second type

We see that B_n^{neo**} is an almost neo balancing number of second type if and only if $8(B_n^{neo**})^2 - 16B_n^{neo**} + 1$ is a perfect square. So we set

$$8(B_n^{neo**})^2 - 16B_n^{neo**} + 1 = y^2,$$

for some positive integer y. Then

$$2[4(B_n^{neo**})^2 - 8B_n^{neo**}] + 1 = y^2,$$

and hence

$$2(2B_n^{neo**} - 2)^2 - 7 = y^2.$$

Taking $x = 2B_n^{neo**} - 2$, we get the Pell equation

$$2x^2 - y^2 = 7. \tag{13}$$

Let Ω^{**} denotes the set of all integer solutions of (13), that is,

$$\Omega^{**} = \{(x, y) : 2x^2 - y^2 = 7\}.$$

Then we can give the following theorem.

Theorem 2.3. *The set of all integer solutions of* (13) *is* $\Omega^{**} = \{(2B_{n-1} + 2C_{n-1}, 8B_{n-1} + C_{n-1}) : n \geq 1\} \cup \{(-2B_n + 2C_n, 8B_n - C_n) : n \geq 1\}$.

Proof. For the Pell equation in (13), the indefinite form is again $F = (2, 0, -1)$. The set of representatives is Rep $= \{[\pm 2 \quad 1]\}$ and $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Here we notice that $[2 \quad 1]M^{n-1}$ generates all integer solutions (x_{2n-1}, y_{2n-1}) and $[2 \quad -1]M^n$ generates all integer solutions (x_{2n}, y_{2n}) for $n \geq 1$. So

$$\begin{aligned} [x_{2n-1} \quad y_{2n-1}] &= [2 \quad 1]M^{n-1} = [2B_{n-1} + 2C_{n-1} \quad 8B_{n-1} + C_{n-1}] \\ [x_{2n} \quad y_{2n}] &= [2 \quad -1]M^n = [-2B_n + 2C_n \quad 8B_n - C_n]. \end{aligned}$$

Thus the set of all integer solutions is $\Omega^{**} = \{(2B_{n-1} + 2C_{n-1}, 8B_{n-1} + C_{n-1}) : n \geq 1\}$ $\cup \{(-2B_n + 2C_n, 8B_n - C_n) : n \geq 1\}$. □

From Theorem 2.3, we can give the following result.

Theorem 2.4. *The general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of second type are*

$$\begin{aligned} B_{2n-1}^{neo**} &= B_{n-1} + C_{n-1} + 1 \\ B_{2n}^{neo**} &= -B_n + C_n + 1 \\ C_{2n-1}^{neo**} &= 8B_{n-1} + C_{n-1} \\ C_{2n}^{neo**} &= 8B_n - C_n \\ R_{2n-1}^{neo**} &= \frac{6B_{n-1} - C_{n-1} - 3}{2} \\ R_{2n}^{neo**} &= \frac{10B_n - 3C_n - 3}{2}, \end{aligned}$$

for $n \geq 1$.

Proof. We proved in Theorem 2.3 that the set of all integer solutions of (13) is $\Omega^{**} = \{(2B_{n-1} + 2C_{n-1}, 8B_{n-1} + C_{n-1}) : n \geq 1\} \cup \{(-2B_n + 2C_n, 8B_n - C_n) : n \geq 1\}$. Since $x = 2B_n^{neo**} - 2$, we get

$$B_{2n-1}^{neo**} = \frac{2B_{n-1} + 2C_{n-1} + 2}{2} = B_{n-1} + C_{n-1} + 1,$$

for $n \geq 1$. Thus from (11), we obtain

$$\begin{aligned} C_{2n-1}^{neo**} &= \sqrt{8(B_{2n-1}^{neo**})^2 - 16B_{2n-1}^{neo**} + 1} \\ &= \sqrt{8(B_{n-1} + C_{n-1} + 1)^2 - 16(B_{n-1} + C_{n-1} + 1) + 1} \\ &= \sqrt{8B_{n-1}^2 + 16B_{n-1}C_{n-1} + 8C_{n-1}^2 - 7} \\ &= \sqrt{C_{n-1}^2 - 1 + 16B_{n-1}C_{n-1} + 8(8B_{n-1}^2 + 1) - 7} \\ &= \sqrt{64B_{n-1}^2 + 16B_{n-1}C_{n-1} + C_{n-1}^2} \\ &= \sqrt{(8B_{n-1} + C_{n-1})^2} \end{aligned}$$

$$= 8B_{n-1} + C_{n-1},$$

for $n \geq 1$. From (10), we deduce that

$$R_{2n-1}^{neo**} = \frac{-2B_{2n-1}^{neo**} - 1 + C_{2n-1}^{neo**}}{2} = \frac{6B_{n-1} - C_{n-1} - 3}{2},$$

for $n \geq 1$. The others can be proved similarly. □

3. Balancing numbers and almost neo balancing numbers

In the previous section, we determined the general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first and second type in terms of balancing and Lucas-balancing numbers. Conversely we can give the general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers in terms of almost neo balancing numbers and almost Lucas-neo balancing numbers of first and second type as follows.

Theorem 3.1. *The general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers are*

$$\begin{aligned} B_n &= \frac{B_{n+1}^{neo*} - 1}{3} \\ b_n &= \frac{-2B_{n+1}^{neo*} + C_{n+1}^{neo*} - 1}{6} \\ C_n &= \frac{C_{n+1}^{neo*}}{3} \\ c_n &= \frac{4B_{n+1}^{neo*} - C_{n+1}^{neo*} - 4}{3}, \end{aligned}$$

for $n \geq 1$, *or*

$$\begin{aligned} B_n &= \frac{B_{2n}^{neo**} + C_{2n}^{neo**} - 1}{7} \\ b_n &= \frac{6B_{2n}^{neo**} - C_{2n}^{neo**} - 13}{14} \\ C_n &= \frac{8B_{2n}^{neo**} + C_{2n}^{neo**} - 8}{7} \\ c_n &= \frac{-4B_{2n}^{neo**} + 3C_{2n}^{neo**} + 4}{7}, \end{aligned}$$

for $n \geq 1$.

Proof. It can be easily derived from Theorems 2.2 and 2.4. □

Thus we construct one-to-one correspondence between all balancing numbers and all almost neo balancing numbers.

4. Relationship with Pell and Pell-Lucas numbers

Recall that general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers can be given in terms of Pell numbers, namely,

$$B_n = \frac{P_{2n}}{2},\ b_n = \frac{P_{2n-1}-1}{2},\ C_n = P_{2n} + P_{2n-1} \text{ and } c_n = P_{2n-1} + P_{2n-2}. \tag{14}$$

Similarly we can give the general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first and of second type in terms of Pell numbers as follows.

Theorem 4.1. *The general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first type are*

$$\begin{aligned} B_n^{neo*} &= \frac{3P_{2n-2}+2}{2},\ n \geq 1, \\ C_n^{neo*} &= 3P_{2n-2} + 3P_{2n-3},\ n \geq 2, \\ R_n^{neo*} &= \frac{3P_{2n-3}-3}{2},\ n \geq 2, \end{aligned}$$

and of second type are

$$\begin{aligned} B_{2n-1}^{neo**} &= \frac{3P_{2n-2}+2P_{2n-3}+2}{2},\ n \geq 2 \\ B_{2n}^{neo**} &= \frac{P_{2n}+2P_{2n-1}+2}{2},\ n \geq 1 \\ C_{2n-1}^{neo**} &= 5P_{2n-2} + P_{2n-3},\ n \geq 2 \\ C_{2n}^{neo**} &= 3P_{2n} - P_{2n-1},\ n \geq 1 \\ R_{2n-1}^{neo**} &= \frac{2P_{2n-2}-P_{2n-3}-3}{2},\ n \geq 2 \\ R_{2n}^{neo**} &= \frac{2P_{2n}-3P_{2n-1}-3}{2},\ n \geq 1. \end{aligned}$$

Proof. Recall that $B_n = \frac{P_{2n}}{2}$ and $b_n = \frac{P_{2n-1}-1}{2}$ by (14). Thus from Theorems 2.2 and 2.4, we get the desired result. □

Conversely, we can give the general terms of even and odd ordered Pell numbers in terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first and of second type as follows.

Theorem 4.2. *The general terms of even and odd ordered Pell numbers are*

$$\begin{aligned} P_{2n} &= \frac{2B_{n+1}^{neo*}-2}{3} \\ P_{2n-1} &= \frac{-2B_{n+1}^{neo*}+C_{n+1}^{neo*}+2}{3}, \end{aligned}$$

for $n \geq 1$, *or*

$$P_{2n} = \frac{2B_{2n}^{neo**}+2C_{2n}^{neo**}-2}{7}$$

$$P_{2n-1} = \frac{6B_{2n}^{neo**} - C_{2n}^{neo**} - 6}{7},$$

for $n \geq 1$.

Proof. Since $P_{2n} = 2B_n$ and $P_{2n-1} = 2b_n + 1$, we get the desired result from Theorem 3.1. □

Thus we construct one-to-one correspondence between Pell numbers and almost neo balancing numbers.

As in (14), we note that the general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers can be given in terms of Pell-Lucas numbers, namely,

$$B_n = \frac{Q_{2n} + Q_{2n-1}}{8},\ b_n = \frac{Q_{2n} - Q_{2n-1} - 4}{8},\ C_n = \frac{Q_{2n}}{2} \text{ and } c_n = \frac{Q_{2n-1}}{2}. \tag{15}$$

Similarly we can give the general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first and of second type in terms of Pell-Lucas numbers as follows.

Theorem 4.3. *The general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first type are*

$$\begin{aligned}
B_n^{neo*} &= \frac{3Q_{2n-2} + 3Q_{2n-3} + 8}{8},\ n \geq 2 \\
C_n^{neo*} &= \frac{3Q_{2n-2}}{2},\ n \geq 1 \\
R_n^{neo*} &= \frac{3Q_{2n-2} - 3Q_{2n-3} - 12}{8},\ n \geq 2,
\end{aligned}$$

and of second type are

$$\begin{aligned}
B_{2n-1}^{neo**} &= \frac{5Q_{2n-2} + Q_{2n-3} + 8}{8},\ n \geq 2 \\
B_{2n}^{neo**} &= \frac{3Q_{2n} - Q_{2n-1} + 8}{8},\ n \geq 1 \\
C_{2n-1}^{neo**} &= \frac{3Q_{2n-2} + 2Q_{2n-3}}{2},\ n \geq 2 \\
C_{2n}^{neo**} &= \frac{Q_{2n} + 2Q_{2n-1}}{2},\ n \geq 1 \\
R_{2n-1}^{neo**} &= \frac{Q_{2n-2} + 3Q_{2n-3} - 12}{8},\ n \geq 2 \\
R_{2n}^{neo**} &= \frac{-Q_{2n} + 5Q_{2n-1} - 12}{8},\ n \geq 1.
\end{aligned}$$

Proof. Applying Theorems 2.2 and 2.4, we get the desired result from (15). □

Conversely, we can give the general terms of even and odd ordered Pell-Lucas numbers in terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first and of second type as follows.

Theorem 4.4. *The general terms of even and odd ordered Pell-Lucas numbers are*

$$Q_{2n} = \frac{2C_{n+1}^{neo*}}{3}$$
$$Q_{2n-1} = \frac{8B_{n+1}^{neo*} - 2C_{n+1}^{neo*} - 8}{3},$$

for $n \geq 1$, *or*

$$Q_{2n} = \frac{16B_{2n}^{neo**} + 2C_{2n}^{neo**} - 16}{7}$$
$$Q_{2n-1} = \frac{-8B_{2n}^{neo**} + 6C_{2n}^{neo**} + 8}{7},$$

for $n \geq 1$.

Proof. Recall that $B_n = \frac{Q_{2n}+Q_{2n-1}}{8}$ and $C_n = \frac{Q_{2n}}{2}$ by (15). Thus from Theorem 3.1, we get the desired result. □

Thus we construct one-to-one correspondence between Pell-Lucas numbers and all almost neo balancing numbers.

5. Relationship with triangular and square triangular numbers

Recall that triangular numbers denoted by T_n are the numbers of the form

$$T_n = \frac{n(n+1)}{2}.$$

It is known that there is a correspondence between balancing (and also cobalancing) numbers and triangular numbers. Indeed from (1), we note that n is a balancing number if and only if n^2 is a triangular number since

$$\frac{(n+r)(n+r+1)}{2} = n^2.$$

So

$$T_{B_n+R_n} = B_n^2. \tag{16}$$

Similarly from (3), n is a cobalancing number if and only if n^2+n is a triangular number since

$$\frac{(n+r)(n+r+1)}{2} = n^2 + n.$$

So

$$T_{b_n+r_n} = b_n^2 + b_n.$$

As in (16), we can give the following theorem.

Theorem 5.1. *B_n^{neo*} is an almost neo balancing number of first type if and only if $(B_n^{neo*})^2 - 2B_n^{neo*} + 2$ is a triangular number, that is,*

$$T_{B_n^{neo*}+R_n^{neo*}} = (B_n^{neo*})^2 - 2B_n^{neo*} + 2,$$

and B_n^{neo**} *is an almost neo balancing number of second type if and only if* $(B_n^{neo**})^2 - 2B_n^{neo**}$ *is a triangular number, that is,*

$$T_{B_n^{neo**}+R_n^{neo**}} = (B_n^{neo**})^2 - 2B_n^{neo**}.$$

Proof. Let n be an almost neo balancing number of first type. Then from (7), we get

$$2n - 1 + nr + \frac{r(r+1)}{2} - \frac{(n-1)n}{2} = 1,$$

and hence

$$\frac{(n+r)(n+r+1)}{2} = n^2 - 2n + 2.$$

Thus

$$T_{B_n^{neo*}+R_n^{neo*}} = (B_n^{neo*})^2 - 2B_n^{neo*} + 2.$$

Similarly, let n be an almost neo balancing number of second type. Then from (7), we get

$$2n - 1 + nr + \frac{r(r+1)}{2} - \frac{(n-1)n}{2} = -1,$$

and hence

$$\frac{(n+r)(n+r+1)}{2} = n^2 - 2n.$$

Thus

$$T_{B_n^{neo**}+R_n^{neo**}} = (B_n^{neo**})^2 - 2B_n^{neo**},$$

as we wanted. □

There are infinitely many triangular numbers that are also square numbers which are called square triangular numbers and is denoted by S_n. Notice that

$$S_n = s_n^2 = \frac{t_n(t_n+1)}{2},$$

where s_n and t_n are the sides of the corresponding square and triangle. We can give the general terms of S_n, s_n and t_n in terms of balancing and cobalancing numbers, namely,

$$S_n = B_n^2, \ s_n = B_n \ \text{ and } \ t_n = B_n + b_n. \tag{17}$$

Their Binet formulas are

$$S_n = \frac{\alpha^{4n} + \beta^{4n} - 2}{32}, \ s_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} \ \text{ and } t_n = \frac{\alpha^{2n} + \beta^{2n} - 2}{4}, \tag{18}$$

for $n \geq 1$. We can give the general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first and second type in terms of s_n and t_n as follows.

Theorem 5.2. *The general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first type are*

$$\begin{aligned} B_n^{neo*} &= 3s_{n-1} + 1, \\ C_n^{neo*} &= 6t_{n-1} + 3, \\ R_n^{neo*} &= -3s_{n-1} + 3t_{n-1}, \end{aligned}$$

for $n \geq 1$, and of second type are

$$\begin{aligned} B_{2n-1}^{neo**} &= s_{n-1} + 2t_{n-1} + 2 \\ B_{2n}^{neo**} &= -s_n + 2t_n + 2 \\ C_{2n-1}^{neo**} &= 8s_{n-1} + 2t_{n-1} + 1 \\ C_{2n}^{neo**} &= 8s_n - 2t_n - 1 \\ R_{2n-1}^{neo**} &= 3s_{n-1} - t_{n-1} - 2 \\ R_{2n}^{neo**} &= 5s_n - 3t_n - 3, \end{aligned}$$

for $n \geq 1$.

Proof. Notice that $s_n = B_n$ and $t_n = B_n + b_n$ by (17). Thus

$$\begin{aligned} t_n &= B_n + b_n \\ &= \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} + \frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2} \\ &= \frac{\alpha^{2n}(1 + \alpha^{-1}) + \beta^{2n}(-1 - \beta^{-1})}{4\sqrt{2}} - \frac{1}{2} \\ &= \frac{\frac{\alpha^{2n} + \beta^{2n}}{2}}{2} - \frac{1}{2} \\ &= \frac{C_n - 1}{2}. \end{aligned}$$

So $C_n = 2t_n + 1$. Applying Theorems 2.2 and 2.4, we get the desired result. □

Conversely, we can give the following theorem.

Theorem 5.3. *The general terms of S_n, s_n and t_n are*

$$\begin{aligned} S_n &= \frac{(B_{n+1}^{neo*})^2 - 2B_{n+1}^{neo*} + 1}{9} \\ s_n &= \frac{B_{n+1}^{neo*} - 1}{3} \\ t_n &= \frac{C_{n+1}^{neo*} - 3}{6}, \end{aligned}$$

for $n \geq 1$, or

$$S_n = \frac{(B_{2n}^{neo**})^2 + (C_{2n}^{neo**})^2 + 2B_{2n}^{neo**}C_{2n}^{neo**} - 2B_{2n}^{neo**} - 2C_{2n}^{neo**} + 1}{49}$$

$$s_n = \frac{B_{2n}^{neo**} + C_{2n}^{neo**} - 1}{7}$$
$$t_n = \frac{8B_{2n}^{neo**} + C_{2n}^{neo**} - 15}{14},$$

for $n \geq 1$.

Proof. From (18), we get

$$\begin{aligned} S_n &= \frac{\alpha^{4n} + \beta^{4n} - 2}{32} \\ &= (\frac{\alpha^{4n} - \beta^{4n}}{4\sqrt{2}})^2 \\ &= \frac{9(\frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}})^2 + 6(\frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}}) + 1 - 6(\frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}}) - 2 + 1}{9} \\ &= \frac{\left[3(\frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}}) + 1\right]^2 - 2\left[3(\frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}}) + 1\right] + 1}{9} \\ &= \frac{(B_{n+1}^{neo*})^2 - 2B_{n+1}^{neo*} + 1}{9}, \end{aligned}$$

by Theorem 2.2. The others can be proved similarly. □

Thus we construct one-to-one correspondence between almost neo balancing numbers and square triangular numbers.

6. Sums of almost neo balancing numbers

Theorem 6.1. *The sum of first n-terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first type are*

$$\sum_{i=1}^{n} B_i^{neo*} = \frac{15B_{n-1} - 3B_{n-2} + 4n - 3}{4}$$
$$\sum_{i=1}^{n} C_i^{neo*} = \frac{21B_{n-1} - 3B_{n-2} + 3}{2}$$
$$\sum_{i=1}^{n} R_i^{neo*} = \frac{3B_{n-1} - 3n + 3}{2},$$

for $n \geq 1$, *and of second type are*

$$\sum_{i=1}^{n} B_i^{neo**} = \begin{cases} 3B_{\frac{n}{2}} + n, \ n \geq 2 \ even \\ \\ 7B_{\frac{n-1}{2}} - B_{\frac{n-3}{2}} + n, \ n \geq 1 \ odd \end{cases}$$
$$\sum_{i=1}^{n} C_i^{neo**} = \begin{cases} 51B_{\frac{n-2}{2}} - 9B_{\frac{n-4}{2}} - 3, \ n \geq 2 \ even \\ \\ 20B_{\frac{n-1}{2}} - 4B_{\frac{n-3}{2}} - 3, \ n \geq 1 \ odd \end{cases}$$

$$\sum_{i=1}^{n} R_i^{neo**} = \frac{1}{2}\begin{cases} 15B_{\frac{n-2}{2}} - 3B_{\frac{n-4}{2}} - 3n - 3, \ n \geq 2 \ even \\ \\ 6B_{\frac{n-1}{2}} - 2B_{\frac{n-3}{2}} - 3n - 3, \ n \geq 1 \ odd. \end{cases}$$

Proof. Recall that $B_1 + B_2 + \cdots + B_n = \frac{5B_n - B_{n-1} - 1}{4}$. Thus Theorem 2.2, we get

$$\begin{aligned} \sum_{i=1}^{n} B_i^{neo*} &= \sum_{i=1}^{n}(3B_{i-1} + 1) \\ &= 3(\sum_{i=1}^{n} B_{i-1}) + n \\ &= 3(\frac{5B_{n-1} - B_{n-2} - 1}{4}) + n \\ &= \frac{15B_{n-1} - 3B_{n-2} + 4n - 3}{4} \end{aligned}$$

for $n \geq 2$. The others can be proved similarly. □

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