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α-Jacobi Type Vector Fields in Hyperbolic 3-space with Natural Statistical Structure

Oleksandr Sadokha and Cengizhan Murathan*

Dedicated to the 65th birthday of Professor Kadri Arslan

(Communicated by Cihan Özgür)

ABSTRACT

In this paper, we fully determine all Jacobi-type vector fields in hyperbolic three-space by taking advantage of both its constant negative curvature and its intrinsic compatibility with statistical manifold structures. The study is a natural extension of the results obtained by Wang and Zhang, [16], under the classical Levi–Civita connection.

Keywords: *Statistical manifolds, statistical models, Jacobi type vector fields, normal distribution, hyperbolic models.*

AMS Subject Classification (2020): *Primary: 53A35 ; Secondary: 53B20.*

1. Introduction and Motivation

1.1. Jacobi field

Let (M, g) be a Riemannian manifold and $\gamma(t)$ be a geodesic on M. A vector field $U(t)$ is said to be Jacobi vector field along γ if it satisfies Jacobi differential equation

$$\frac{D^2}{dt^2}U(t) + \mathring{R}(U(t), \gamma'(t))\gamma'(t) = 0,$$

where $\frac{D}{dt}$ denotes covariant derivative along the geodesic $\gamma(t)$ and $\mathring{R}$ is Riemannian curvature tensor field of the manifold M. As follows from the Jacobi equation, Jacobi vector fields naturally arise in the study of variations of geodesics and offer valuable insight into the intrinsic geometry of Riemannian manifolds. In particular, they describe the behavior of nearby geodesics and provide a useful framework for understanding the divergence or convergence of geodesic flows, depending on the curvature of the manifold. As a natural generalization of a Jacobi type vector field on a Riemannian manifold M is defined as follows:

$$\mathring{\nabla}_X \mathring{\nabla}_X U + \mathring{R}(U, X)X = 0,$$

for any vector field X. In [6], Deshmukh introduced an alternative notion of Jacobi vector fields, defined by the equation

$$\mathring{\nabla}_X \mathring{\nabla}_X U - \mathring{\nabla}_{\mathring{\nabla}_X X} U + \mathring{R}(U, X)X = 0. \tag{1.1}$$

The above definition generalizes the classical Jacobi equation along geodesics, reducing to it when X is the tangent vector field of a geodesic. Using the linearity of the curvature operator, the sum of Deshmukh-type Jacobi terms along all orthonormal directions can be written in terms of the Ricci tensor

$$\mathring{\triangle} U + \mathring{R}ic(U) = 0,$$

where $\mathring{R}ic(U) = \sum_{i=1}^{n} \mathring{R}(U, e_i)e_i$, $\mathring{\triangle}$ is Laplace-Beltrami operator and $\{e_i : 1 \leq i \leq n\}$ is a local orthonormal frame of M (see [6]).

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By polarization, Deshmukh also proved in [6] that if U is a Jacobi type vector field, then for any vector field X and Y on M, the following relation holds:

$$\mathring{\nabla}_X\mathring{\nabla}_Y U - \mathring{\nabla}_{\mathring{\nabla}_X Y}U + \mathring{R}(U,X)Y = 0.$$

Motivated by the above definition, Wang and Zhang [16] investigated Jacobi-type vector fields on the real space forms, namely the hyperbolic 3-space H^3 and the Euclidean 3-space $\mathbb{R}^3$.

1.2. Statistical model

Let

$$S = S(x,\theta) = \{p(x,\theta) : \int_\chi p(x,\theta) = 1, p(x,\theta) > 0, \theta = (\theta_1,...,\theta_n) \in \Xi \subset \mathbb{R}^n\}$$

be a family probability distribution. Here $p(x,\theta)$ denotes the probability density function of the data set x and Ξ is referred to as the parameter space. Consider one-to-one mapping

$$\phi : S(x,\theta) \to \Xi \subset \mathbb{R}^n;\ p(x,\theta) \to \theta$$

such that $Rank\phi = dim\Xi = n$. In this setting, $S(x,\theta)$ is parametrized by (Ξ,ϕ) and is refeered to an n-dimensional statistical model [2] .

For a statistical model $S(x,\theta)$, the Fisher information matrix is defined

$$g^F_{ij}(\theta) = \int_{x\in\chi} \frac{\partial \log p(x,\theta)}{\partial\theta_i}\frac{\partial \log p(x,\theta)}{\partial\theta_j}p(x,\theta)dx$$

with χ denoting sample space. Since the matrix $g^F(\theta)$ is symmetric, positive definite and non-degenerate, it defines a Riemannian metric on statistical models, which can be locally expreesed as

$$g^F = \sum_{i,j=1}^{n} g^F_{ij}(\theta)d\theta_i d\theta_j$$

commonly referred to as the Fisher metric [2].

For a fixed constant $\alpha \in \mathbb{R}$, the Christoffel symbols of a torsion-free α-connection ∇^α on $S(x,\theta)$ are defined by

$$\begin{aligned}\Gamma^\alpha_{ij,k} &= \int_{x\in\chi}\{\frac{\partial^2 \log p(x,\theta)}{\partial\theta_i\partial\theta_j} + \frac{1-\alpha}{2}(\frac{\partial \log p(x,\theta)}{\partial\theta_j})(\frac{\partial \log p(x,\theta)}{\partial\theta_j})\}\\ &\quad\times(\frac{\partial \log p(x,\theta)}{\partial\theta_k})p(x,\theta))dx.\end{aligned}$$

Moreover, the following equation holds

$$\partial\theta_i g^F(\partial\theta_j,\partial\theta_k) = g^F(\nabla^\alpha_{\partial\theta_i}\partial\theta_j,\partial\theta_k) + g^F(\partial\theta_j,\nabla^{(-\alpha)}_{\partial\theta_i}\partial\theta_k).$$

In information geometry, the α-connection ∇^α is referred as Amari-Chentsov α-connection. Hence triplet $(\nabla^\alpha, \nabla^{(-\alpha)}, g^F)$ is called α-statistical structure on $S(x,\theta)$.

A fundamental example of a statistical model is the multivariate normal (Takano-Gaussian) distribution. It is parameterized by $\theta = (\mu_1,..,\mu_i,..,\mu_n,\sigma) \in \mathbb{R}^n \times \mathbb{R}^+ = \tilde{H}^{n+1}_F$, where $(\mu_1,..,\mu_i,..,\mu_n)$ denotes mean value, σ^2 represents variance. The corresponding probability density function is given by

$$p(x,\theta) = \frac{1}{\sqrt[n]{2\pi\sigma^2}}\prod_{i=1}^{n}\exp\{-\frac{(x-\mu_i)^2}{2\sigma^2}\},\ x\in\mathbb{R}^n.$$

By a straightforward computation, the Fisher information metric,

$$g^F = \frac{1}{\sigma^2}(2nd\sigma^2 + \sum_{i=1}^{n} d\mu_i^2), \tag{1.2}$$

[5], [7], [15]. The corresponding Amari-Chentsov α-connection coefficients, [7], are given by

$$\begin{aligned} {}^F\nabla^{\alpha}_{\partial\mu_i}\partial\mu_j &= \frac{1-\alpha}{2n\sigma}\delta_{ij}\partial\sigma, {}^F\nabla^{\alpha}_{\partial\mu_i}\partial\sigma = {}^F\nabla^{\alpha}_{\partial\sigma}\partial\mu_i = -\frac{1+\alpha}{\sigma}\partial\mu_i, \\ {}^F\nabla^{\alpha}_{\partial\sigma}\partial\sigma &= -\frac{1+2\alpha}{\sigma}\partial\sigma. \end{aligned} \tag{1.3}$$

The above statistical model $(H^{n+1}_F, g^F, {}^F\nabla)$ is a space of constant curvature $-\frac{1-\alpha^2}{2n}$ and is known Takano Gaussian space [7].

1.3. Statistical manifold

Let (M, g) be a Riemannian manifold and let ∇ be torsion free connection. If the tensor

$$Q(X, Y, Z) = (\nabla_X g)(Y, Z)$$

is totally symmetric for all vector fields X, Y, Z on M then the triple (M, g, ∇) is called a statistical manifold (see [2], [8]).

The difference between ∇ and $\mathring{\nabla}$ connections on M can be expressed as a $(1, 2)-$type tensor field $\mathcal{K}$, defined by

$$\mathcal{K} = \nabla - \mathring{\nabla}. \tag{1.4}$$

Since ∇ and $\mathring{\nabla}$ are torsion-free we have

$$\mathcal{K}_X Y = \mathcal{K}_Y X, \quad g(\mathcal{K}_X Y, Z) = g(Y, \mathcal{K}_X Z) \tag{1.5}$$

for any X, Y, Z vector fields on M. In the notion of a statistical manifold (M, g, ∇), the tensor field $\mathcal{K}$ is related to the cubic form Q by

$$Q(X, Y, Z) = 2g(\mathcal{K}_X Y, Z).$$

The conjugate (or dual) connection ∇^* of an affine connection ∇ with respect to a Riemannian metric g is defined by the identity on M

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y), \tag{1.6}$$

for any X, Y, Z vector fields on M ([10], [14]). It follows from (1.4) and (1.6) that

$$2\mathcal{K} = \nabla - \nabla^* \text{ and } \mathring{\nabla} = \frac{\nabla + \nabla^*}{2}.$$

It follows immediately that $(\nabla^*_X g)(Y, Z) = -Q(X, Y, Z)$. Therefore, (M, g, ∇) is a statistical manifold if and only if (M, g, ∇^*) is also a statistical manifold.

On a statistical manifold (M, g, ∇), the curvature tensors of the connection ∇ and its dual connection ∇^* are defined, respectively, by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and

$$R^*(X, Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z.$$

These tensors satisfy the following duality relation for all vector fields X, Y, Z, W on M :

$$g(R(X, Y)Z, W) = -g(X, R^*(Y, Z)W).$$

The curvature tensor associated with an affine connection ∇ can be related to the Riemannian curvature tensor $\mathring{R}$, for any vector fields X, Y, Z on M, by the following identity

$$R(X, Y)Z = \mathring{R}(X, Y)Z + \frac{1}{4}[K_X, K_Y]Z - \frac{1}{2}((\nabla_X K)(Y, Z) - (\nabla_Y K)(X, Z)), \tag{1.7}$$

[11]. Let ∇ and ∇^* be affine connections on a Riemannian manifold (M, g) that are conjugate with respect to the metric tensor field g. For any real number α, one can define the $\alpha-$connection ∇^α by the formula

$$\nabla^\alpha = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*. \tag{1.8}$$

From this definition, it follows directly that ∇^α and $\nabla^{-\alpha}$ are conjugate with respect to g for every $\alpha \in \mathbb{R}$.

The curvature of R^α with respect to ∇^α is given by

$$R^\alpha(X,Y)Z = \frac{1+\alpha}{2}R^*(X,Y)Z + \frac{1-\alpha}{2}R(X,Y)Z + \frac{1-\alpha^2}{4}(K_Y K_X Z - K_X K_Y Z) \tag{1.9}$$

for any vector fields X, Y, Z on M [1], [17].

Statistical manifolds constitute the basic geometric framework of information geometry, providing a differential–geometric interpretation of statistical models, divergence functions and parameter estimation. In this setting, a statistical manifold can be viewed as a Riemannian manifold in which each point represents a probability distribution, thereby establishing a natural connection between information geometry and (affine) differential geometry [2], [8], [10].

Now let us consider upper half space hyperbolic model:

$$H^{n+1} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

equipped with the Riemannian metric

$$g_{H^{n+1}} = \frac{dx_1^2 + \cdots + dx_{n+1}^2}{x_{n+1}^2}.$$

Now we define a similarity map

$$\psi : H_F^{n+1} \to H^{n+1}, \psi(\mu_1, \mu_2, ..., \mu_n, \sigma) = (\frac{\mu_1}{\sqrt{2n}}, \frac{\mu_2}{\sqrt{2n}}, ..., \frac{\mu_n}{\sqrt{2n}}, \sigma),$$

which establishes a correspondence between the Fisher information manifold of the Takano Gaussian space H_F^{n+1} and the upper half-space model of hyperbolic space H^{n+1}. By taking the pullback with respect to ψ, we obtain $g^F = 2n\psi^*(\tilde{g}_{\tilde{H}^{n+1}})$, where ψ^* denotes pullback map induced by ψ. Hence, the Fisher geometry of the Takano Gaussian space is equivalent, up to scaling, to the geometry of the $(n+1)$-dimensional hyperbolic space in the upper half-space representation [5].

Lauritzen [8] posed the natural question of whether every statistical manifold can be realized as a statistical model. This problem became one of the central questions in the development of information geometry and remained open for a long time. It was later resolved in the affirmative by Lê in [9] . The main theorem of Lê can be stated as follows:

Theorem 1.1 ([9]). *Any smooth statistical manifold (M^n, g, T) can be immersed into the statistical manifold (Cap_+^N, g^F, T^{A-C}) for some number N. Hence any statistical manifold is a statistical model. Here Cap_+^N denotes all positive probability distributions on sample space Ω^n.*

Let (M, g, ∇^α) be an statistical manifold. Consider a smooth one-parameter family of geodesics $\gamma_s(t) = \gamma(s,t)$ on M, parametrized by s with $\gamma_0(t) = \gamma(t)$ denoting reference geodesic. We define the variation vector field along γ by

$$U(t) = \frac{\partial}{\partial s} \mid_{s=0} \gamma_s(t),$$

which describes the infinitesimal displacement of nearby geodesics. The second covariant derivative of U along γ can be written as

$$\nabla^\alpha_{\dot\gamma}\nabla^\alpha_{\dot\gamma} U = \nabla^\alpha_{\partial_t}\nabla^\alpha_{\partial_t}(\partial_s\gamma_s) \mid_{s=0} .$$

By the torsion-free property of ∇^α the order of covariant derivatives can be interchanged according to

$$\nabla^\alpha_{\partial_t}\nabla^\alpha_{\partial_s} X - \nabla^\alpha_{\partial_s}\nabla^\alpha_{\partial_t} X = R^\alpha(\partial_t\gamma_s, \partial_s\gamma_s)X,$$

where R^α is the curvature tensor of ∇^α and $X = \partial_t\gamma_s$. Since each γ_s is a geodesic, it satisfies $\nabla^\alpha_{\partial_t}\partial_t\gamma_s = 0$ which implies $\nabla^\alpha_{\partial_s}\nabla^\alpha_{\partial_t}\partial_t\gamma_s = 0$. Combining these observations, one obtains $\nabla^\alpha_{\partial_t}\nabla^\alpha_{\partial_t}(\partial_s\gamma_s) \mid_{s=0} = R^\alpha(\dot\gamma, U)\dot\gamma$. Consequently, the variation vector field U satisfies the Jacobi equation

$$\nabla^\alpha_{\dot\gamma}\nabla^\alpha_{\dot\gamma} U + R^\alpha(U, \dot\gamma)\dot\gamma = 0.$$

In [3], Arif et al. investigated the geometric structures of the Freund manifold by defining a Riemannian metric and computing the associated curvature tensor and scalar curvature. They further analyzed the Jacobi

fields to study the instability of geodesics from a differential geometric perspective. Moreover, by considering the probability density function of the two-dimensional Freund distribution as a statistical manifold, they defined the corresponding Riemannian metric, the α-connection, and the associated α-curvature. The behavior of Jacobi fields and the convergence of geodesics were also discussed, providing insights into the geometric foundations of information geometry.

Furthermore, Cafaro and Ali [4] conducted a comparative geometric analysis of the entropic dynamical models $ED1$ and $ED2$. In particular, they examined the evolution of Jacobi fields and the geodesic deviation equations under the Fisher–Rao metric. Their results reveal that the $ED1$ model, involving the exponential distribution, exhibits stronger geodesic divergence, reflecting higher dynamical instability of the associated flow. In contrast, the $ED2$ manifold, constructed as the product of two normal distributions, displays smoother curvature and weaker geodesic deviation, indicating a more stable geometric structure. These findings emphasize the interplay between statistical structure and geometric stability in the framework of entropic dynamics.

Similar to the equation (1.1) it is possible to define Jacobi-type vector fields on a statistical manifold (M, g) with respect to the Amari-Chentsov connection ∇^α . In this case, $\alpha-$Jacobi-type vector fields can be defined, and the equation (1.1) takes the form

$$\nabla^\alpha_X \nabla^\alpha_Y U - \nabla^\alpha_{\nabla^\alpha_X Y} U + R^\alpha(U, X)Y = 0, \forall X, Y, U \in \Gamma(TM). \tag{1.10}$$

If $\alpha = 0$ then $\alpha-$Jacobi-type vector fields returns to the standart Jacobi type vector fields.

On the other hand, Wang and Zhang [16] investigated the same problem on the simplest non-compact real space forms, namely the hyperbolic three-space H^3 and the Euclidean three-space $\mathbb{R}^3$ and determined all Jacobi-type vector fields on these spaces.

Motivated by these studies, we consider the hyperbolic three-space H^3, which possesses an intrinsic statistical manifold structure, and focus on determining all Jacobi-type vector fields with respect to the Amari-Chentsov $\alpha-$connection ∇^α.

2. $\alpha-$Jacobi type vector fields in upper half-space model of hyperbolic space $\tilde{\mathbf{H}}^3$ with natural statistical structure

For $n = 2$ we consider homothety map

$$\psi : H^3_F \to H^3, \psi(\mu_1, \mu_2, \sigma) = (\frac{\mu_1}{2}, \frac{\mu_2}{2}, \sigma), \tag{2.1}$$

so that $\mu_1 = 2x, \mu_2 = 2y$ and $\sigma = z$. Then the corresponding derivative map

$$d\psi : T_{(\mu_1,\mu_2,\sigma)} H^3_F \longrightarrow T_{(x,y,z)} H^3$$

induces the following transformation on the corresponding tangent space basis vectors:

$$d\psi(\partial\mu_1) = 2\partial x, d\psi(\partial\mu_2) = 2\partial y \text{ and } d\psi(\partial_\sigma) = \partial z. \tag{2.2}$$

Now, we consider the equation (1.3) for $n = 2$ on H^3_F . This means that components of the α-connection ${}^F\nabla^\alpha$ are given by

$$\begin{array}{lll} {}^F\nabla^\alpha_{\partial\mu_1}\partial\mu_1 = \frac{1-\alpha}{4\sigma}\partial\sigma, & {}^F\nabla^\alpha_{\partial\mu_2}\partial\mu_1 = 0, & {}^F\nabla^\alpha_{\partial\sigma}\partial\mu_1 = -\frac{1+\alpha}{\sigma}\partial\mu_1, \\ {}^F\nabla^\alpha_{\partial\mu_1}\partial\mu_2 = 0, & {}^F\nabla^\alpha_{\partial\mu_2}\partial\mu_2 = \frac{1-\alpha}{4\sigma}\partial\sigma, & {}^F\nabla^\alpha_{\partial\sigma}\partial\mu_2 = -\frac{1+\alpha}{\sigma}\partial\mu_2, \\ {}^F\nabla^\alpha_{\partial\mu_1}\partial\sigma = -\frac{1+\alpha}{\sigma}\partial\mu_1, & {}^F\nabla^\alpha_{\partial\mu_2}\partial\sigma = -\frac{1+\alpha}{\sigma}\partial\mu_2, & {}^F\nabla^\alpha_{\partial\sigma}\partial\sigma = -\frac{1+2\alpha}{\sigma}\partial\sigma. \end{array} \tag{2.3}$$

By using the equations (2.2) and (2.3), we can define corresponding torsion free α connections on H^3. The components are given by

$$\begin{array}{lll} \nabla^\alpha_{\partial x}\partial x = \frac{1-\alpha}{z}\partial z, & \nabla^\alpha_{\partial y}\partial x = 0, & \nabla^\alpha_{\partial z}\partial\mu_x = -\frac{1+\alpha}{z}\partial x, \\ \nabla^\alpha_{\partial x}\partial y = 0, & \nabla^\alpha_{\partial y}\partial y = \frac{1-\alpha}{z}\partial z, & \nabla^\alpha_{\partial z}\partial y = -\frac{1+\alpha}{z}\partial y, \\ \nabla^\alpha_{\partial x}\partial z = -\frac{1+\alpha}{z}\partial x, & \nabla^\alpha_{\partial y}\partial z = -\frac{1+\alpha}{z}\partial y, & \nabla^\alpha_{\partial z}\partial z = -\frac{1+2\alpha}{z}\partial z. \end{array} \tag{2.4}$$

Similarly, the components of the $(-\alpha)$-connection are obtained as

$$\begin{array}{lll} \nabla^{(-\alpha)}_{\partial x}\partial x = \frac{1+\alpha}{z}\partial z, & \nabla^{(-\alpha)}_{\partial y}\partial x = 0, & \nabla^{(-\alpha)}_{\partial z}\partial \mu_x = -\frac{1-\alpha}{z}\partial x, \\ \nabla^{(-\alpha)}_{\partial x}\partial y = 0, & \nabla^{(-\alpha)}_{\partial y}\partial y = \frac{1+\alpha}{z}\partial z, & \nabla^{(-\alpha)}_{\partial z}\partial y = -\frac{1-\alpha}{z}\partial y, \\ \nabla^{(-\alpha)}_{\partial x}\partial z = -\frac{1-\alpha}{z}\partial x, & \nabla^{(-\alpha)}_{\partial y}\partial z = -\frac{1-\alpha}{z}\partial y, & \nabla^{(-\alpha)}_{\partial z}\partial z = -\frac{1-2\alpha}{z}\partial z. \end{array} \tag{2.5}$$

It can be easily verified that $\partial\eta_i g(\partial\eta_j, \partial\eta_k) = g(\nabla^{\alpha}_{\partial\eta_i}\partial\eta_j, \partial\eta_k) + g(\partial\eta_j, \nabla^{(-\alpha)}_{\partial\eta_i}\partial\eta_k)$ where $(\eta_1, \eta_2, \eta_3) = (x, y, z)$. Hence, by the means of (2.1), a natural statistical structure arises on the three-dimensional hyperbolic space H^3.

Now we give main theorem of this study.

Theorem 2.1. *On the hyperbolic 3-space H^3 endowed with the statistical structure $(g, \nabla^{\alpha}, \nabla^{(-\alpha)})$, a vector field $V = \sum_{i=1}^{3} f_i e_i$ is a $(\alpha)-$ Jacobi-type vector field if and only if the component functions $f_i(x, y, z)$ are given by*

$f_1(x, y, z) = (b_6 y + \frac{1}{1+\alpha} b_9 x + b_{11})\frac{1}{z}$, $f_2(x, y, z) = (-b_6 x + \frac{1}{1+\alpha} b_9 y + b_{13})\frac{1}{z}$

and $f_3(x, y, z) = \frac{1}{1+\alpha} b_9$, where the coefficienties b_i are constant, $\alpha \neq -1, 0, 1$ and $\{e_1 = z\partial x, e_2 = z\partial y, e_3 = z\partial z\}$ is a local orthonormal frame field of H^3.

Proof. Suppose that $V = f_1 e_1 + f_2 e_2 + f_3 e_3$ be a $\alpha-$Jacobian-type vector field on H^3. Here, f_1, f_2 and f_3 are differentiable functions on H^3 and $e_1 = z\partial x$, $e_2 = z\partial y$ and $e_3 = z\partial z$. By the equations (1.10) and (2.4) we compute

$$\nabla^{\alpha}_{e_1}\nabla^{\alpha}_{e_1} V - \nabla^{\alpha}_{\nabla^{\alpha}_{e_1} e_1} V + R^{\alpha}(V, e_1)\, e_1 =$$

$$\begin{aligned} = & \left(z^2 \frac{\partial^2 f_1}{\partial x^2} - 2(1+\alpha) z \frac{\partial f_3}{\partial x} - (1-\alpha) z \frac{\partial f_1}{\partial z} - (1-\alpha) f_1 \right) e_1 + \\ & + \left(z^2 \frac{\partial^2 f_2}{\partial x^2} - (1-\alpha) z \frac{\partial f_2}{\partial z} - (1-\alpha) f_2 \right) e_2 + \\ & + \left(z^2 \frac{\partial^2 f_3}{\partial x^2} + 2(1-\alpha) z \frac{\partial f_1}{\partial x} - (1-\alpha) z \frac{\partial f_3}{\partial z} - 2(1-\alpha) f_3 \right) e_3 \\ = & \ 0, \end{aligned} \tag{2.6}$$

$$\nabla^{\alpha}_{e_2}\nabla^{\alpha}_{e_2} V - \nabla^{\alpha}_{\nabla^{\alpha}_{e_2} e_2} V + R^{\alpha}(V, e_2)\, e_2 =$$

$$\begin{aligned} & \left(z^2 \frac{\partial^2 f_1}{\partial y^2} - (1-\alpha) z \frac{\partial f_1}{\partial z} - (1-\alpha) f_1 \right) e_1 + \\ & + \left(z^2 \frac{\partial^2 f_2}{\partial y^2} - 2(1+\alpha) z \frac{\partial f_3}{\partial y} - (1-\alpha) z \frac{\partial f_2}{\partial z} - (1-\alpha) f_2 \right) e_2 + \\ & + \left(z^2 \frac{\partial^2 f_3}{\partial y^2} + 2(1-\alpha) z \frac{\partial f_2}{\partial y} - (1-\alpha) z \frac{\partial f_3}{\partial z} - 2(1-\alpha) f_3 \right) e_3 \\ = & \ 0, \end{aligned} \tag{2.7}$$

$$\nabla^{\alpha}_{e_3}\nabla^{\alpha}_{e_3} V - \nabla^{\alpha}_{\nabla^{\alpha}_{e_3} e_3} V + R^{\alpha}(V, e_3)\, e_3 =$$

$$\begin{aligned} & \left(z^2 \frac{\partial^2 f_1}{\partial z^2} + z \frac{\partial f_1}{\partial z} - f_1 \right) e_1 + \left(z^2 \frac{\partial^2 f_2}{\partial z^2} + z \frac{\partial f_2}{\partial z} - f_2 \right) e_2 + \\ & + \left(z^2 \frac{\partial^2 f_3}{\partial z^2} + (1-2\alpha) z \frac{\partial f_3}{\partial z} \right) e_3 \\ = & \ 0, \end{aligned} \tag{2.8}$$

$$\nabla^{\alpha}_{e_1}\nabla^{\alpha}_{e_2} V - \nabla^{\alpha}_{\nabla^{\alpha}_{e_1} e_2} V + R^{\alpha}(V, e_1)\, e_2 =$$

$$\begin{aligned}&\left(z^2\frac{\partial^2 f_1}{\partial x\partial y}-(1+\alpha)\,z\frac{\partial f_3}{\partial y}\right)e_1+\left(z^2\frac{\partial^2 f_2}{\partial x\partial y}-(1+\alpha)\,z\frac{\partial f_3}{\partial x}\right)e_2+\\&+\left(z^2\frac{\partial^2 f_3}{\partial x\partial y}+(1-\alpha)\,z\left(\frac{\partial f_1}{\partial y}+\frac{\partial f_2}{\partial x}\right)\right)e_3\\=\;&0,\end{aligned}\tag{2.9}$$

$$\nabla^{\alpha}_{e_3}\nabla^{\alpha}_{e_1}V-\nabla^{\alpha}_{\nabla^{\alpha}_{e_3}e_1}V+R^{\alpha}\left(V,e_3\right)e_1=$$

$$\begin{aligned}&\left(z^2\frac{\partial^2 f_1}{\partial x\partial z}-(1+\alpha)\,z\frac{\partial f_3}{\partial z}+z\frac{\partial f_1}{\partial x}\right)e_1+\left(z^2\frac{\partial^2 f_2}{\partial x\partial z}+z\frac{\partial f_2}{\partial x}\right)e_2+\\&+\left(z^2\frac{\partial^2 f_3}{\partial x\partial z}+(1-\alpha)\,z\left(\frac{\partial f_1}{\partial z}+\frac{\partial f_3}{\partial x}\right)+(1-\alpha)\,f_1\right)e_3\\=\;&0,\end{aligned}\tag{2.10}$$

and

$$\nabla^{\alpha}_{e_3}\nabla^{\alpha}_{e_2}V-\nabla^{\alpha}_{\nabla^{\alpha}_{e_3}e_2}V+R^{\alpha}\left(V,e_3\right)e_2=$$

$$\begin{aligned}&\left(z^2\frac{\partial^2 f_1}{\partial y\partial z}+z\frac{\partial f_1}{\partial y}\right)e_1+\left(z^2\frac{\partial^2 f_2}{\partial y\partial z}-(1+\alpha)\,z\frac{\partial f_3}{\partial z}+z\frac{\partial f_2}{\partial y}\right)e_2+\\&+\left(z^2\frac{\partial^2 f_3}{\partial y\partial z}+(1-\alpha)\,z\left(\frac{\partial f_2}{\partial z}+\frac{\partial f_3}{\partial y}\right)+(1-\alpha)\,f_2\right)e_3\\=\;&0.\end{aligned}\tag{2.11}$$

As the remaining three equations are of the same form as those previously presented, they are not considered further here.

According to the equations (2.6)-(2.11) V is an α Jacobi-type vector field if and only if the following eighteen partial differential equations are satisfied:

$$z^2\frac{\partial^2 f_1}{\partial x^2}-2\,(1+\alpha)\,z\frac{\partial f_3}{\partial x}-(1-\alpha)\,z\frac{\partial f_1}{\partial z}-(1-\alpha)\,f_1=0,\tag{2.12}$$

$$z^2\frac{\partial^2 f_2}{\partial x^2}-(1-\alpha)\,z\frac{\partial f_2}{\partial z}-(1-\alpha)\,f_2=0,\tag{2.13}$$

$$z^2\frac{\partial^2 f_3}{\partial x^2}+2\,(1-\alpha)\,z\frac{\partial f_1}{\partial x}-(1-\alpha)\,z\frac{\partial f_3}{\partial z}-2\,(1-\alpha)\,f_3=0,\tag{2.14}$$

$$z^2\frac{\partial^2 f_1}{\partial y^2}-(1-\alpha)\,z\frac{\partial f_1}{\partial z}-(1-\alpha)\,f_1=0,\tag{2.15}$$

$$z^2\frac{\partial^2 f_2}{\partial y^2}-2\,(1+\alpha)\,z\frac{\partial f_3}{\partial y}-(1-\alpha)\,z\frac{\partial f_2}{\partial z}-(1-\alpha)\,f_2=0,\tag{2.16}$$

$$z^2\frac{\partial^2 f_3}{\partial y^2}+2\,(1-\alpha)\,z\frac{\partial f_2}{\partial y}-(1-\alpha)\,z\frac{\partial f_3}{\partial z}-2\,(1-\alpha)\,f_3=0,\tag{2.17}$$

$$z^2\frac{\partial^2 f_1}{\partial z^2}+z\frac{\partial f_1}{\partial z}-f_1=0,\tag{2.18}$$

$$z^2\frac{\partial^2 f_2}{\partial z^2}+z\frac{\partial f_2}{\partial z}-f_2=0,\tag{2.19}$$

$$z^2\frac{\partial^2 f_3}{\partial z^2}+(1-2\alpha)\,z\frac{\partial f_3}{\partial z}=0,\tag{2.20}$$

and

$$z^2\frac{\partial^2 f_1}{\partial x\partial y}-(1+\alpha)\,z\frac{\partial f_3}{\partial y} = 0, \tag{2.21}$$

$$z^2\frac{\partial^2 f_2}{\partial x\partial y}-(1+\alpha)\,z\frac{\partial f_3}{\partial x} = 0, \tag{2.22}$$

$$z^2\frac{\partial^2 f_3}{\partial x\partial y}+(1-\alpha)\,z\left(\frac{\partial f_1}{\partial y}+\frac{\partial f_2}{\partial x}\right) = 0, \tag{2.23}$$

$$z^2\frac{\partial^2 f_1}{\partial x\partial z}-(1+\alpha)\,z\frac{\partial f_3}{\partial z}+z\frac{\partial f_1}{\partial x} = 0, \tag{2.24}$$

$$z^2\frac{\partial^2 f_2}{\partial x\partial z}+z\frac{\partial f_2}{\partial x} = 0, \tag{2.25}$$

$$z^2\frac{\partial^2 f_3}{\partial x\partial z}+(1-\alpha)\,z\left(\frac{\partial f_1}{\partial z}+\frac{\partial f_3}{\partial x}\right)+(1-\alpha)\,f_1 = 0, \tag{2.26}$$

$$z^2\frac{\partial^2 f_2}{\partial y\partial z}-(1+\alpha)\,z\frac{\partial f_3}{\partial z}+z\frac{\partial f_2}{\partial y} = 0, \tag{2.27}$$

$$z^2\frac{\partial^2 f_1}{\partial y\partial z}+z\frac{\partial f_1}{\partial y} = 0, \tag{2.28}$$

$$z^2\frac{\partial^2 f_3}{\partial y\partial z}+(1-\alpha)\,z\left(\frac{\partial f_2}{\partial z}+\frac{\partial f_3}{\partial y}\right)+(1-\alpha)\,f_2 = 0. \tag{2.29}$$

It is sufficient to solve this system of differential equations to find the V $\alpha-$Jacobi-type vector field. Using (2.18) and (2.19) we obtain

$$f_1(x,y,z)=\frac{1}{z}H_3(x,y)+zH_4(x,y) \tag{2.30}$$

and

$$f_2(x,y,z)=\frac{1}{z}H_5(x,y)+zH_6(x,y), \tag{2.31}$$

respectively. Here $H_i(x,y)$ are the smooth functions depending only on the variables x and y. Substituting (2.30) into (2.28) and (2.31) into (2.25), we obtain $H_4(x,y)=l_1(x)$ and $H_6(x,y)=l_2(y)$, where l_1 and l_2 are arbitrary smooth functions depending only on x and y, respectively. Thus, equations (2.30) and (2.31) take the following form:

$$f_1(x,y,z) = \frac{1}{z}H_3(x,y)+zl_1(x), \tag{2.32}$$

$$f_2(x,y,z) = \frac{1}{z}H_5(x,y)+zl_2(y). \tag{2.33}$$

Substituting equations (2.32) into (2.15), (2.33) into (2.13) and recalling $z>0$ we have

$$f_1(x,y,z) = \left((1-\alpha)\,y^2\frac{1}{z}+z\right)l_1(x)+y\frac{1}{z}l_3(x)+\frac{1}{z}l_4(x), \tag{2.34}$$

$$f_2(x,y,z) = \left((1-\alpha)\,x^2\frac{1}{z}+z\right)l_2(y)+x\frac{1}{z}l_5(y)+\frac{1}{z}l_6(y). \tag{2.35}$$

Here l_3, l_4 represent arbitrary smooth functions of x, whereas and l_5, l_6 denote arbitrary smooth functions of y.

By solving equation (2.20), we obtain

$$f_3(x,y,z)=\frac{1}{2\alpha}z^{2\alpha}H_1(x,y)+H_2(x,y), \tag{2.36}$$

where H_1 and H_2 are smooth functions depending only on x, y. If we substitute equations (2.34) and (2.36) into (2.24) and equations (2.35) and (2.36) into (2.27), we get the following relations, respectively,

$$H_1(x,y) = \frac{2}{(1+\alpha)}z^{2(1-\alpha)}\frac{\partial l_1(x)}{\partial x}, \tag{2.37}$$

$$H_1(x,y) = \frac{2}{(1+\alpha)}z^{2(1-\alpha)}\frac{\partial l_2(y)}{\partial y}. \tag{2.38}$$

From (2.37) and (2.38) we have

$$l_1(x) = b_1 x + b_2, \tag{2.39}$$
$$l_2(y) = b_1 y + b_3. \tag{2.40}$$

Putting (2.39) into (2.37) or (2.40) into (2.38) we obtain

$$H_1(x,y) = \frac{2}{(1+\alpha)} z^{2(1-\alpha)} b_1.$$

Thus, equation (2.36) becomes

$$f_3(x,y,z) = \frac{1}{\alpha(1+\alpha)} z^2 b_1 + H_2(x,y). \tag{2.41}$$

Substituting (2.34), (2.41) into equation (2.21) and (2.35), (2.41) into equation (2.22), yields the following relations:

$$(1+\alpha)\frac{\partial H_2(x,y)}{\partial y} = \frac{\partial l_3(x)}{\partial x} + 2(1-\alpha) y b_1, \tag{2.42}$$
$$(1+\alpha)\frac{\partial H_2(x,y)}{\partial x} = \frac{\partial l_5(y)}{\partial y} + 2(1-\alpha) x b_1. \tag{2.43}$$

Differentiating the equation (2.42) with respect to x and equation (2.43) with respect to y, we have

$$\frac{\partial^2 l_3(x)}{\partial x^2} = \frac{\partial^2 l_5(y)}{\partial y^2}.$$

In this case there exist a constant $b_4 \in \mathbb{R}$

$$\frac{\partial^2 l_3(x)}{\partial x^2} = b_4 = \frac{\partial^2 l_5(y)}{\partial y^2}.$$

Solving the above equation we obtain

$$l_3(x) = x^2 b_4 + x b_5 + b_6, \tag{2.44}$$
$$l_5(y) = y^2 b_4 + y b_7 + b_8, \tag{2.45}$$

where b_6 and b_8 are constants.

Now putting (2.44) and (2.45) into (2.42) and (2.43) respectively after straightforward calculations, we find

$$H_2(x,y) = \frac{1-\alpha}{1+\alpha}\left(x^2+y^2\right) b_1 + 2\frac{1}{1+\alpha} xy b_4 + \frac{1}{1+\alpha} y b_5 + \frac{1}{1+\alpha} x b_7 + \frac{1}{1+\alpha} b_9 \tag{2.46}$$

with b_9 being a constant.

By substituting equations (2.39) and (2.44) into (2.34), equations (2.40) and (2.45) into (2.35), and equation (2.46) into (2.41), we obtain

$$\begin{aligned} f_1(x,y,z) &= (1-\alpha) x y^2 \frac{1}{z} b_1 + x z b_1 + (1-\alpha) y^2 \frac{1}{z} b_2 \\ &\quad + z b_2 + x^2 y \frac{1}{z} b_4 + x y \frac{1}{z} b_5 + y \frac{1}{z} b_6 + \frac{1}{z} l_4(x), \end{aligned} \tag{2.47}$$

$$\begin{aligned} f_2(x,y,z) &= (1-\alpha) x^2 y \frac{1}{z} b_1 + y z b_1 + (1-\alpha) x^2 \frac{1}{z} b_3 \\ &\quad + z b_3 + x y^2 \frac{1}{z} b_4 + x y \frac{1}{z} b_7 + x \frac{1}{z} b_8 + \frac{1}{z} l_6(y), \end{aligned} \tag{2.48}$$

$$\begin{aligned} f_3(x,y,z) &= \frac{1-\alpha}{1+\alpha}\left(x^2+y^2+\frac{1}{\alpha(1-\alpha)} z^2\right) b_1 + 2\frac{1}{1+\alpha} xy b_4 \\ &\quad + \frac{1}{1+\alpha} y b_5 + \frac{1}{1+\alpha} x b_7 + \frac{1}{1+\alpha} b_9. \end{aligned} \tag{2.49}$$

Now putting (2.47) and (2.49) into (2.12) and after some along calculations we obtain

$$l_4(x) = (1-\alpha)x^3b_1 + (1-\alpha)x^2b_2 + x^2yb_4 + x^2b_7 + xb_{10} + b_{11}.$$

Using this result, we then find

$$\begin{aligned} f_1(x,y,z) &= (1-\alpha)x^3\frac{1}{z}b_1 + (1-\alpha)xy^2\frac{1}{z}b_1 + xzb_1 \\ &\quad + (1-\alpha)\left(x^2+y^2\right)\frac{1}{z}b_2 + zb_2 + 2x^2y\frac{1}{z}b_4 \\ &\quad + xy\frac{1}{z}b_5 + y\frac{1}{z}b_6 + x^2\frac{1}{z}b_7 + x\frac{1}{z}b_{10} + \frac{1}{z}b_{11}, \end{aligned} \tag{2.50}$$

where b_{10} and b_{11} are constants.

Similarly by substituting (2.48) and (2.49) into (2.16) and after some straightforward calculations, we obtain

$$l_6(y) = (1-\alpha)y^3b_1 + (1-\alpha)y^2b_3 + xy^2b_4 + y^2b_5 + yb_{12} + b_{13}.$$

Using this result, the function $f_2(x,y,z)$ is expressed as follows:

$$\begin{aligned} f_2(x,y,z) &= (1-\alpha)y^3\frac{1}{z}b_1 + (1-\alpha)x^2y\frac{1}{z}b_1 + yzb_1 \\ &\quad + (1-\alpha)\left(x^2+y^2\right)\frac{1}{z}b_3 + zb_3 + 2xy^2\frac{1}{z}b_4 + y^2\frac{1}{z}b_5 + \\ &\quad xy\frac{1}{z}b_7 + x\frac{1}{z}b_8 + y\frac{1}{z}b_{12} + \frac{1}{z}b_{13}, \end{aligned} \tag{2.51}$$

where b_{12} and b_{13} are constants.

Putting (2.49), (2.50) into (2.26) and (2.49), (2.51) into (2.29), respectively we obtaing the following polynomial equations :

$$4b_1x + 2b_4y + 2(1+\alpha)b_2 + b_7 = 0, \tag{2.52}$$

$$4b_1y + 2b_4x + 2(1+\alpha)b_3 + b_5 = 0. \tag{2.53}$$

In this case we have $b_1 = b_4 = 0$, $b_7 = -2(1+\alpha)b_2$ and $b_5 = -2(1+\alpha)b_3$. In light of these results, the functions f_1, f_2 and f_3 reduce to the following equations:

$$\begin{aligned} f_1(x,y,z) &= (1-\alpha)\left(x^2+y^2\right)\frac{1}{z}b_2 + zb_2 - 2(1+\alpha)b_3xy\frac{1}{z} \\ &\quad + y\frac{1}{z}b_6 - 2(1+\alpha)b_2x^2\frac{1}{z} + x\frac{1}{z}b_{10} + \frac{1}{z}b_{11}, \end{aligned} \tag{2.54}$$

$$\begin{aligned} f_2(x,y,z) &= (1-\alpha)b_3\left(x^2+y^2\right)\frac{1}{z} + b_3z - 2(1+\alpha)b_3y^2\frac{1}{z} \\ &\quad - 2(1+\alpha)b_2xy\frac{1}{z} + b_8x\frac{1}{z} + b_{12}y\frac{1}{z} + b_{13}\frac{1}{z}, \end{aligned} \tag{2.55}$$

$$f_3(x,y,z) = -2b_3y - 2b_2x + \frac{1}{1+\alpha}b_9. \tag{2.56}$$

By subtracting equation (2.14) from equation (2.17), the following equation is obtained.

$$z^2\left(\frac{\partial^2 f_3}{\partial x^2} - \frac{\partial^2 f_3}{\partial y^2}\right) + 2(1-\alpha)z\left(\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y}\right) = 0. \tag{2.57}$$

Substituting the last smooth functions f_1, f_2 and f_3 into (2.57), we get following polynomial equation:

$$-4\alpha b_2x + 4\alpha b_3y + b_{10} - b_{12} = 0. \tag{2.58}$$

Since $\alpha \neq 0$, we have $b_2 = b_3 = 0$ and $b_{10} = b_{12}$. Substituting these obtained values into equations (2.54), (2.55), and (2.56), we find

$$f_1(x,y,z) = b_6\frac{y}{z} + b_{10}\frac{x}{z} + b_{11}\frac{1}{z}, \tag{2.59}$$

$$f_2(x,y,z) = b_8\frac{v}{z} + b_{10}\frac{y}{z} + b_{13}\frac{1}{z}, \tag{2.60}$$

$$f_3(x,y,z) = \frac{1}{1+\alpha}b_9. \tag{2.61}$$

Now putting (2.59), (2.60) and (2.61) into (2.14) and (2.23), repectively we have

$$b_9 = (1+\alpha)b_{10} \text{ and } b_6 = -b_8. \tag{2.62}$$

Finally, this completes the proof of the theorem. □

Now we discuss for special coefficients $\alpha = 1, -1$.

Theorem 2.2. *On the hyperbolic 3-space H^3 equipped with the statistical structure $(g, \nabla^1, \nabla^{(-1)})$, a vector field $V = \sum_{i=1}^{3} f_i e_i$ is a $(1)-$ Jacobi-type vector field on H^3 if and only if its the component functions $f_i(x,y,z)$ are expressed as follows:*

$$f_1(x,y,z) = 2C_1xy\frac{1}{z} + 2C_2x^2\frac{1}{z} + C_5y\frac{1}{z} + 2C_3xz + C_6x\frac{1}{z} + C_7\frac{1}{z} + C_8z,$$

$$f_2(x,y,z) = 2C_1y^2\frac{1}{z} + 2C_2xy\frac{1}{z} + 2C_3yz + C_9x\frac{1}{z} + C_{10}y\frac{1}{z} + C_{11}z + C_{12}\frac{1}{z}$$

and

$$f_3(x,y,z) = C_1y + C_2x + C_3z^2 + C_4,$$

where the coefficients C_i are arbitary constant.

Proof. From (2.12)-(2.29) we have following equations:

$$z\frac{\partial^2 f_1}{\partial x^2} - 4\frac{\partial f_3}{\partial x} = 0, \tag{2.63}$$

$$\frac{\partial^2 f_2}{\partial x^2} = 0, \tag{2.64}$$

$$\frac{\partial^2 f_3}{\partial x^2} = 0, \tag{2.65}$$

$$\frac{\partial^2 f_1}{\partial y^2} = 0, \tag{2.66}$$

$$z\frac{\partial^2 f_2}{\partial y^2} - 4\frac{\partial f_3}{\partial y} = 0, \tag{2.67}$$

$$\frac{\partial^2 f_3}{\partial y^2} = 0, \tag{2.68}$$

$$z^2\frac{\partial^2 f_1}{\partial z^2} + z\frac{\partial f_1}{\partial z} - f_1 = 0, \tag{2.69}$$

$$z^2\frac{\partial^2 f_2}{\partial z^2} + z\frac{\partial f_2}{\partial z} - f_2 = 0, \tag{2.70}$$

$$z\frac{\partial^2 f_3}{\partial z^2} - \frac{\partial f_3}{\partial z} = 0, \tag{2.71}$$

and

$$z\frac{\partial^2 f_1}{\partial x\partial y}-2\frac{\partial f_3}{\partial y} = 0, \tag{2.72}$$
$$z\frac{\partial^2 f_2}{\partial x\partial y}-2\frac{\partial f_3}{\partial x} = 0, \tag{2.73}$$
$$\frac{\partial^2 f_3}{\partial x\partial y} = 0, \tag{2.74}$$
$$z\frac{\partial^2 f_1}{\partial x\partial z}-2\frac{\partial f_3}{\partial z}+\frac{\partial f_1}{\partial x} = 0, \tag{2.75}$$
$$z\frac{\partial^2 f_2}{\partial x\partial z}+\frac{\partial f_2}{\partial x} = 0, \tag{2.76}$$
$$\frac{\partial^2 f_3}{\partial x\partial z} = 0, \tag{2.77}$$
$$z\frac{\partial^2 f_2}{\partial y\partial z}-2\frac{\partial f_3}{\partial z}+\frac{\partial f_2}{\partial y} = 0, \tag{2.78}$$
$$z\frac{\partial^2 f_1}{\partial y\partial z}+\frac{\partial f_1}{\partial y} = 0, \tag{2.79}$$
$$\frac{\partial^2 f_3}{\partial y\partial z} = 0. \tag{2.80}$$

By the equations (2.66), (2.64) and (2.68), we obtain, respectively,

$$f_1(x,y,z) = yA_1(x,z)+B_1(x,z), \tag{2.81}$$
$$f_2(x,y,z) = xA_2(y,z)+B_2(y,z), \tag{2.82}$$
$$f_3(x,y,z) = yA_3(x,z)+B_3(x,z), \tag{2.83}$$

where A_i and B_i are differentiable functions of two variables. By substituting (2.83) into (2.65), (2.71), (2.74), (2.77) and (2.80),we have

$$f_3(x,y,z) = yC_1+xC_2+z^2C_3+C_4, \tag{2.84}$$

where the coefficients C_1, C_2, C_3 and C_4 are arbitrary constants.
By substituting (2.81) and (2.84) into (2.72) we obtain that the smooth function A_1 has the form

$$A_1(x,z) = 2x\frac{1}{z}C_1+k_1(z),$$

where k_1 is an arbitrary differentiable function of z. Therefore, the equation (2.81) can be written as

$$f_1(x,y,z) = 2C_1xy\frac{1}{z}+k_1(z)\,y+B_1(x,z). \tag{2.85}$$

Using (2.85) and (2.79) we have

$$k_1(z) = \frac{C_5}{z},$$

where C_5 is a constant. Therefore, (2.85) can be rewritten in the form

$$f_1(x,y,z) = \frac{2}{z}xyC_1+y\frac{1}{z}C_5+B_1(x,z). \tag{2.86}$$

Next, substituting (2.86) into (2.63), we have

$$B_1(x,z) = 2x^2\frac{1}{z}C_2+xk_2(z)+k_3(z),$$

where k_1 and k_2 are arbitrary differentiable function of z. This gives us

$$f_1(x,y,z) = 2xy\frac{1}{z}C_1+y\frac{1}{z}C_5+2x^2\frac{1}{z}C_2+xk_2(z)+k_3(z). \tag{2.87}$$

By substituting the equations (2.87) and (2.84) into (2.75) we obtain

$$k_2(z) = 2C_3 z + \frac{1}{z}C_6$$

and consequently

$$f_1(x,y,z) = 2xy\frac{1}{z}C_1 + y\frac{1}{z}C_5 + 2x^2\frac{1}{z}C_2 + 2xzC_3 + x\frac{1}{z}C_6 + k_3(z). \tag{2.88}$$

Finally the using the equations (2.69) and (2.88) we have

$$k_3(z) = \frac{1}{z}C_7 + zC_8.$$

Therefore, the function f_1 takes the form

$$f_1(x,y,z) = 2xy\frac{1}{z}C_1 + y\frac{1}{z}C_5 + 2x^2\frac{1}{z}C_2 + 2xzC_3 + x\frac{1}{z}C_6 + \frac{1}{z}C_7 + zC_8,$$

where the coefficients C_5, C_6, C_7 and C_8 are arbitary constants.
Similarly, by using the equations (2.67), (2.70), (2.73), (2.76) and (2.78) we obtain

$$f_2(x,y,z) = 2xy\frac{1}{z}C_2 + x\frac{1}{z}C_9 + 2y^2\frac{1}{z}C_1 + 2yzC_3 + y\frac{1}{z}C_{10} + zC_{11} + \frac{1}{z}C_{12},$$

where C_9, C_{10}, C_{11} and C_{12} are arbitary constants. So we obtain our results. □

By substituting $\alpha = -1$ into the equations (2.12)-(2.29) and solving the resulting system of differential equations, we obtain the following theorem.

Theorem 2.3. *On the hyperbolic 3-space H^3 equipped with the statistical structure $(g, \nabla^1, \nabla^{(-1)})$, a vector field $V = \sum_{i=1}^{3} f_i e_i$ is a $(-1)-$ Jacobi-type vector field if and only if the component functions $f_i(x,y,z)$ are given by*

$$\begin{aligned} f_1(x,y,z) &= D_3 x\frac{1}{z} + D_5 y\frac{1}{z} + (D_4 + D_6)\frac{1}{z}, \\ f_2(x,y,z) &= D_8 x\frac{1}{z} + D_3 y\frac{1}{z} + (D_9 + D_{11})\frac{1}{z} \end{aligned}$$

and

$$f_3(x,y,z) = -2(D_5 + D_8)xy\frac{1}{z^2} + D_{14}\frac{1}{z^2} + D_3,$$

where the coefficienties D_i are arbitary constants.

For $\alpha = 0$, we give following remark.

Remark 2.1. Wang and Zhang obtained equation (3.1) in [16] in order to compute Jacobi vector fields in the 3-dimensional hyperbolic space H^3 with respect to the Levi–Civita connection. Nevertheless, the system they employ necessarily yields nine additional compatibility equations, which do not appear in their formulation.

These missing equations are the following:

$$\begin{aligned}
z\frac{\partial^2 f_1}{\partial x\partial y}-\frac{\partial f_3}{\partial y} &= 0,\\
z\frac{\partial^2 f_2}{\partial x\partial y}-\frac{\partial f_3}{\partial x} &= 0,\\
z\frac{\partial^2 f_3}{\partial x\partial y}+\left(\frac{\partial f_1}{\partial y}+\frac{\partial f_2}{\partial x}\right) &= 0,\\
z\frac{\partial^2 f_1}{\partial x\partial z}-\frac{\partial f_3}{\partial z}+\frac{\partial f_1}{\partial x} &= 0,\\
z\frac{\partial^2 f_2}{\partial x\partial z}+\frac{\partial f_2}{\partial x} &= 0,\\
z^2\frac{\partial^2 f_3}{\partial x\partial z}+z\left(\frac{\partial f_1}{\partial z}+\frac{\partial f_3}{\partial x}\right)+f_1 &= 0,\\
z\frac{\partial^2 f_2}{\partial y\partial z}-\frac{\partial f_3}{\partial z}+\frac{\partial f_2}{\partial y} &= 0,\\
z\frac{\partial^2 f_1}{\partial y\partial z}+\frac{\partial f_1}{\partial y} &= 0,\\
z^2\frac{\partial^2 f_3}{\partial y\partial z}+z\left(\frac{\partial f_2}{\partial z}+\frac{\partial f_3}{\partial y}\right)+f_2 &= 0.
\end{aligned}$$

When these equations are taken into account, the parameters appearing in the general solution of the system are forced to satisfy $k_4=0, k_5=0$ and $k_{11}=-k_9$. Hence the components of any Jacobi vector field on H^3 reduce to the following simplified form:

$$\begin{aligned}
f_1(x,y,z) &= \frac{1}{2z}k_6\left(x^2-y^2\right)+\frac{1}{z}k_7xy+\frac{1}{z}k_8x+\frac{1}{z}k_9y+\frac{1}{z}k_{10}-\frac{1}{2}k_6z,\\
f_2(x,y,z) &= \frac{1}{z}k_6xy+\frac{1}{2z}k_7\left(y^2-x^2\right)+\frac{1}{z}k_8y-\frac{1}{z}k_9x+\frac{1}{z}k_{12}-\frac{1}{2}k_7z,\\
f_3(x,y,z) &= k_6x+k_7y+k_8.
\end{aligned}$$

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