Some recent results on Lebesgue-Ramanujan-Nagell type equations

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Abstract

Let d and δ be fixed positive integers. Consider the Diophantine equation $x^2 + d^s = \delta y^n$ where x, y, n and s are nonnegative integer unknowns. This equation is usually called the generalized Lebesgue-Ramanujan-Nagell equation. It has a long history and rich content. Recently, a survey paper on the generalized Lebesgue-Ramanujan-Nagell equation has been written by M.-H. Le and G. Soydan, [6].

Denote by $h = h(-p)$ the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$ with p prime. It is well known that $h = 1$ for $p \in \{3, 7, 11, 19, 43, 67, 163\}$. Recently, all the solutions of the Diophantine equation $x^2 + p^s = 4y^n$ with $h = 1$ were given by Chakraborty et al. in [3]. In this talk, we consider the Diophantine equation $x^2 + p^s = 2^r y^n$ in unknown integers (x, y, s, r, n) where $s \geq 0$, $r \geq 3$, $n \geq 3$, $h \in \{1, 2, 3\}$ and $\gcd(x, y) = 1$. Our main tools include the known results from the modularity of Galois representations associated with Frey-Hellegoaurch elliptic curves (i.e. modular approach) [1], the symplectic method [4], a Thue-Mahler solver which was improved by Gherga and Siksek [5] in `MAGMA` [2] and elementary methods of classical algebraic number theory. This work was supported by the Research Fund of Bursa Uludağ University under Project No: FGA-2023-1545. This is a joint work with Elif Kızıldere Mutlu, [7, 8].

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